Variations on inductive-recursive definitions

Fredrik Nordvall Forsberg

University of Strathclyde, Glasgow

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Joint work with Neil Ghani, Conor McBride, Peter Hancock and Stephan Spahn

An inductive definition

```
data Rose (A : Set) : Set where
  leaf : Rose A
  node : A → List (Rose A) → Rose A
```

We can represent Rose A by a functor $F_{\text{Rose}} : \text{Set} \to \text{Set}$:

$$F_{\text{Rose}}(X) = 1 + A \times \text{List}\, X$$

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$$F_{\mathsf{Rose}}(X) = 1 + A \times \mathsf{List}\, X$$

Rose A is the initial algebra of F_{Rose}.

An inductive-recursive definition

A universe closed inder $\mathbb{N}$ and Σ.

```
data U : Set
T : U → Set

data U where
  nat : U
  sig   : (a : U) → (b : T a → U) → U

T nat = ℕ
T (sig a b) = Σ (T a) (T ∘ b)
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U and T defined simultaneously.

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U and *T* defined simultaneously.

Also (U, T) is the initial algebra of a functor.

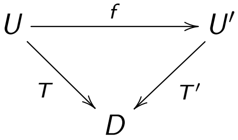
Category of families of Ds

The category Fam D for $D : \mathrm{Set}_1$:

- objects pairs (U, T) where

$$U : \mathrm{Set}$$
$$T : U \to D$$

- morphisms $(U, T) \to (U', T')$ are $f : U \to U'$ s.t.



commutes.

Note: Fam : Cat $\to$ Cat is a monad; D considered as discrete category.

An endofunctor on Fam Set

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is represented by F : Fam Set → Fam Set where

$$F(X, Q) = (1, \_ \mapsto \mathbb{N}) + \left(\left(\Sigma a : X\right)(Q\, a \to X), (a, b) \mapsto \Sigma\ (Q\, a)\ (Q \circ b)\right)$$

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(U, T) is the initial algebra of F.

Representing inductive definitions

Not every functor defines a datatype. We want our inductive definitions to be strictly positive.

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We can codify such definitions as follows (baby Dybjer-Setzer [1999, 2003, 2006]):

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data ID : Set₁ where
  stop : ID
  side : (A : Set) → (c : A → ID) → ID
  ind  : (A : Set) → (c : ID) → ID
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Each code gives rise to a functor:

$$
\begin{aligned}
&[\![-]\!] : \mathrm{ID} \to (\mathrm{Set} \to \mathrm{Set})\\
&[\![\mathrm{stop}]\!]\ X = 1\\
&[\![\mathrm{side}\ A\ c]\!]\ X = \big(\Sigma x : A\big)([\![c\ x]\!]\ X)\\
&[\![\mathrm{ind}\ A\ c]\!]\ X = (A \to X) \times [\![c]\!]\ X
\end{aligned}
$$

A code for List A

$$
\begin{array}{lll}
\mathsf{stop} & : & \mathsf{ID} \\
\mathsf{side} & : & (\mathtt{A} : \mathsf{Set}) \to \\
 & & (\mathtt{c} : \mathtt{A} \to \mathsf{ID}) \to \mathsf{ID} \\
\mathsf{ind} & : & (\mathtt{A} : \mathsf{Set}) \to \\
 & & (\mathtt{c} : \mathsf{ID}) \to \mathsf{ID}
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The datatype

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data List (A : Set) : Set where
  [] : List A
  _::_ : A → List A → List A
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is represented by

$$c_{\mathsf{List}} = \mathsf{side}\,\{'[], '{::}\}\,('[] \mapsto\ \mathsf{stop};\ '{::} \mapsto \mathsf{side}\ A\ (\_ \mapsto \mathsf{ind}\ 1\ \mathsf{stop}))$$

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Note: $\mathsf{side}\,\{\mathsf{tag}_c, \mathsf{tag}_d\}\,(\mathsf{tag}_c \mapsto c;\ \mathsf{tag}_d \mapsto d)$ for encoding coproducts of codes.

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data DS (D E : Set₁) : Set₁ where
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data DS (D E : Set₁) : Set₁ where
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$$[\![\_]\!] : \mathsf{DS}\ D\ E \to \mathsf{Fam}\ D \to \mathsf{Fam}\ E$$

coproducts in Fam D

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\end{aligned}$$

Note: Fam $1 \cong$ Set and DS $1\ 1 \cong$ ID.

A code for a universe

The code

$$c_{\Sigma\mathbb{N}} = \sigma\ \{\mathsf{nat}, \mathsf{sig}\}\ (\mathsf{nat} \mapsto \iota\ \mathbb{N};$$
$$\mathsf{sig} \mapsto \delta\ 1\ (X \mapsto (\delta\ (X\,\star)\ (Y \mapsto \iota\ (\Sigma\ (X\,\star)\ Y)))))$$

represents $F : \mathsf{Fam\ Set} \to \mathsf{Fam\ Set}$ where

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Closure under composition?

DS codes represent functors; are they closed under composition?

That is, given $c : \mathsf{DS}\ C\ D$ and $d : \mathsf{DS}\ D\ E$, is there a code $d \bullet c : \mathsf{DS}\ C\ E$ representing $[\![d]\!] \circ [\![c]\!] : \mathsf{Fam}\ C \to \mathsf{Fam}\ E$?

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- Solve $F(G(X)) \cong X$, not just $F(X) \cong X$. E.g. $c_{\mathsf{Rose}} = c_{\mathsf{List}} \bullet c_{\mathsf{List}}$.
- Longer term goal: want syntax-independent characterisation of induction-recursion (cf polynomial functors [Gambino and Kock]) — will likely be closed under composition.

A proof attempt

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But what about δ? (So far, we can compose with constant functors. . .)

Composing with δ

$$[\![\delta\ A\ F]\!]_0([\![c]\!]_0 Z) = \big(\Sigma g : A \to [\![c]\!]_0 Z\big)\big([\![F\big([\![c]\!]_1(Z) \circ g\big)]\!]_0([\![c]\!] Z)\big)$$

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Progress could be made if we had

1. $A \longrightarrow c$
2. "Concatenation" of codes

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Spoiler alert: these are also necessary conditions.

“Concatenation” of codes

Item 2 is easy, because DS D is a monad (Ghani and Hancock [2016]):

Proposition. *There is an operation*

$$\_ \gg= \_ : DS\ C\ D \to (D \to DS\ C\ E) \to DS\ C\ E$$

such that $[\![c \gg= g]\!]\ Z \cong [\![c]\!]\ Z \gg=_{Fam} (e \mapsto [\![g\ e]\!]\ Z)$.

□

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such that $[\![c \gg= g]\!]\ Z \cong [\![c]\!]\ Z \gg=_{\mathit{Fam}} (e \mapsto [\![g\ e]\!]\ Z)$.
Concretely,

$$[\![c \gg= g]\!]_0\ Z = \left(\Sigma x : [\![c]\!]_0\ Z\right) [\![g\,([\![c]\!]_1\ Z\,x)]\!]_0\ Z$$
$$[\![c \gg= g]\!]_1\ Z\,(x, y) = [\![g([\![c]\!]_1\ Z\,x)]\!]_1\ Z\,y$$

□

Trying to define $S \longrightarrow c$

This time ι and δ are easy, but:

$$\begin{aligned} S \to [\![\sigma\ A\ f]\!]_0\, Z &= S \to \big(\Sigma a : A\big)([\![f\ a]\!]_0\, Z) \\ &\cong \big(\Sigma g : S \to A\big)\big((x : S) \to [\![f\ (g\ x)]\!]_0\, Z\big) \end{aligned}$$

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To continue inductively, we need to generalise to a dependent product

$$\pi : (S : \mathsf{Set}) \to (S \to \mathsf{DS}\ D\ E) \to \mathsf{DS}\ D\ E$$

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To continue inductively, we need to generalise to a dependent product

$$\pi : \big(S : \mathsf{Set}\big) \to (S \to \mathsf{DS}\ D\ E) \to \mathsf{DS}\ D\ E$$

But we cannot define this because we have nothing to induct on anymore.

Powers from composition

In fact, any definition of composition would give us powers:

Theorem. *A composition operator*

$$\_ \bullet \_ : DS\ D\ E \to DS\ C\ D \to DS\ C\ E$$

is definable if and only if a power operator

$$\_ \longrightarrow \_ : (S : Set) \to DS\ D\ E \to DS\ D\ (S \to E)$$

is definable. □

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is definable. □

This (apparent) lack of powers thus suggests that DS, as an axiomatisation of a class of functors, could perhaps be improved upon.

Variations on inductive-recursive definitions

This leads us to investigate alternative classes of functors axiomatising inductive-recursive definitions.

If one wants closure under composition, two natural options suggest themselves:

1. Restrict dependency so that $S \longrightarrow c$ is definable $\rightsquigarrow$ uniform codes (Peter Hancock).

2. Add a π combinator to the system $\rightsquigarrow$ polynomial codes (Conor McBride).

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Take-home message: There are many axiomatisations of induction-recursion.

Uniform codes

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Originally due to Peter Hancock (2012).

Discovered while trying to define composition for DS.

Uniformity by associating like in the 60s

In

$$\sigma : (A : \mathsf{Set}) \to (c : A \to \mathsf{DS}\,DE) \to \mathsf{DS}\,DE$$

nonuniformity comes from c depending on A.

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Consequence: the code c for "the rest of the constructor" is always of the same "shape".

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$$\sigma : (A : \mathsf{Set}) \to (c : A \to \mathsf{DS}\,DE) \to \mathsf{DS}\,DE$$

nonuniformity comes from c depending on A.

Idea: Instead make A depend on (the information in) c.

Consequence: the code c for "the rest of the constructor" is always of the same "shape".

Left-nested instead of right-nested (Pollack: Dependently Typed Records in Type Theory [2002]).

Uniform codes UF

Let $D, E : \mathsf{Set}_1$. $\mathsf{Uni}\ D : \mathsf{Set}_1$ and $\mathsf{Info} : \mathsf{Uni}\ D \to \mathsf{Set}_1$ are inductive-recursively given by

$$\begin{aligned}
\iota_{\mathsf{UF}} &: \mathsf{Uni}\ D \\
\sigma_{\mathsf{UF}} &: (c : \mathsf{Uni}\ D) \to (A : \mathsf{Info}\ c \to \mathsf{Set}) \to \mathsf{Uni}\ D \\
\delta_{\mathsf{UF}} &: (c : \mathsf{Uni}\ D) \to (A : \mathsf{Info}\ c \to \mathsf{Set}) \to \mathsf{Uni}\ D
\end{aligned}$$

$$\begin{aligned}
\mathsf{Info}\ \iota_{\mathsf{UF}} &= \mathbf{1} \\
\mathsf{Info}\ (\sigma_{\mathsf{UF}}\ c\ A) &= (\Sigma\gamma : \mathsf{Info}\ c)(A\ \gamma) \\
\mathsf{Info}\ (\delta_{\mathsf{UF}}\ c\ A) &= (\Sigma\gamma : \mathsf{Info}\ c)(A\ \gamma \to D)
\end{aligned}$$

Large set of uniform codes $\mathsf{UF}\ D\ E = (\Sigma c : \mathsf{Uni}\ D)(\mathsf{Info}\ c \to E)$.

Decoding uniform codes

$$[\![\_]\!]_{\mathsf{Uni}} : \mathsf{Uni}\ D \to \mathsf{Fam}\ D \to \mathsf{Set}$$

$$[\![\_]\!]_{\mathsf{Info}} : (c : \mathsf{Uni}\ D) \to (Z : \mathsf{Fam}\ D) \to [\![\ c\ ]\!]_{\mathsf{Uni}}\ Z \to \mathsf{Info}\ c$$

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$$\llbracket \iota_{\mathsf{UF}} \rrbracket_{\mathsf{Uni}}\ (U, T) = 1$$
$$\llbracket \sigma_{\mathsf{UF}}\ c\ A \rrbracket_{\mathsf{Uni}}\ (U, T) = \big(\Sigma x : \llbracket c \rrbracket_{\mathsf{Uni}}\ (U, T)\big)(A(\llbracket c \rrbracket_{\mathsf{Info}}\ (U, T)\ x))$$
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$$\vdots$$

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Finally for $(c, \alpha) : \mathsf{UF}\ D\ E = (\Sigma c : \mathsf{Uni}\ D)(\mathsf{Info}\ c \to E)$

$$\llbracket (c, \alpha) \rrbracket = (\llbracket c \rrbracket_{\mathsf{Uni}}\ -, \alpha \circ \llbracket c \rrbracket_{\mathsf{Info}}\ -) : \mathsf{Fam}\ D \to \mathsf{Fam}\ E$$

A code for W-types

```
data W (S : Set)(P : S → Set) : Set where
  sup : (s : S) → (P s → W S P) → W S P
```

$$c_{\mathsf{W}\ S\ P,\mathsf{UF}} = \delta_{\mathsf{UF}}\ \big(\sigma_{\mathsf{UF}}\ \iota_{\mathsf{UF}}\ (\_ \mapsto S)\big)\ ((\_, s) \mapsto (P\ s)) : \mathsf{Uni}\ 1$$

$$[\![\ c_{\mathsf{W}\ S\ P,\mathsf{UF}}\ ]\!]_{\mathsf{Uni}}\,(U, T) = \big(\Sigma(\star, s) : 1 \times S\big)(P(s) \to U)$$

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$$[\![c_{\mathsf{W}\ S\ P,\mathsf{DS}}]\!]_0\ (U, T) = \big(\Sigma s : S\big)\big(\Sigma f : (P(s) \to U)\big)1$$

Coproducts of uniform codes

A priori we do not longer have coproducts of codes — DS coproducts relied exactly on non-uniformity of σ.

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Proposition. *For every uniform code c,* $[\![c]\!] Z \cong [\![\sigma_{\mathsf{UF}}\, c\, (\_ \mapsto 1)]\!] Z$ *and* $[\![c]\!] Z \cong [\![\delta_{\mathsf{UF}}\, c\, (\_ \mapsto 0)]\!] Z$. □

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E.g.

$$\sigma_{\mathsf{UF}}\, (\delta_{\mathsf{UF}}\, \iota_{\mathsf{UF}}\, A)\, B +_{\mathsf{UF}} \delta_{\mathsf{UF}}\, \iota_{\mathsf{UF}}\, A' = \sigma_{\mathsf{UF}}\, (\delta_{\mathsf{UF}}\, (\sigma_{\mathsf{UF}}\, \iota_{\mathsf{UF}}\, 2)\, [A, A'])\, [B, 0]$$

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Theorem. $[\![c +_{\mathsf{UF}} d]\!] Z \cong [\![c]\!] Z + [\![d]\!] Z$. □

UF $\hookrightarrow$ DS

Since uniform codes are "backwards", we can translate UF to DS the same way one reverses a list using an accumulator:

$$\mathsf{accUFtoDS} : (c : \mathsf{Uni}\ D) \to (\mathsf{Info}\ c \to \mathsf{DS}\ D\ E) \to \mathsf{DS}\ D\ E$$

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defined by

$$\begin{aligned}
\mathsf{accUFtoDS}\ \iota_{\mathsf{UF}}\ F &= F\ \star \\
\mathsf{accUFtoDS}\ (\sigma_{\mathsf{UF}}\ c\ A)\ F &= \mathsf{accUFtoDS}\ c\ (\gamma \mapsto \sigma\ (A\ \gamma)\ (a \mapsto F\ (\gamma, a))) \\
\mathsf{accUFtoDS}\ (\delta_{\mathsf{UF}}\ c\ A)\ F &= \mathsf{accUFtoDS}\ c\ (\gamma \mapsto \delta\ (A\ \gamma)\ (h \mapsto F\ (\gamma, h)))
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\end{aligned}$$

Proposition. $[\![\mathit{accUFtoDS}\ c\ (\iota \circ \alpha)]\!]\ Z \cong [\![\ (c, \alpha)\ ]\!]\ Z.$ □

Going the other way seems unlikely.

Consequences for soundness

This means that UF can piggyback on Dybjer and Setzer [1999]'s proof of existence of initial algebras.

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However the construction of (Uni, Info) itself is one instance of large induction-recursion, albeit a particularly simple instance. No additional assumptions are needed in the set-theoretical model.

UF is not a monad

We have gained uniformity, which makes powers definable.

Unfortunately, the uniformity also means that we no longer have a monad.

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Bind should graft trees, but grafting a collection of uniform trees might not result in a uniform tree.

Towards composition: combined bind and powers

Is all lost? No. We can still define the instance of bind that we need, combined with a power operation. (Note: only the set depends on Info c.)

$$- \gg\!\!=\![- \longrightarrow -] : (c : \mathsf{Uni}\ D) \to (\mathsf{Info}\ c \to \mathsf{Set}) \to \mathsf{Uni}\ D \to \mathsf{Uni} D$$

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As usual, we need to define this simultaneously with its meaning on Info:

$$(c \gg\!\!= [E \longrightarrow d])_{\mathsf{Info}} : \mathsf{Info}\ (c \gg\!\!= [E \longrightarrow d]) \to \big(\Sigma x : \mathsf{Info}\ c\big)(E\, x \to \mathsf{Info}\ d)$$

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Proposition. *There is an equivalence*

$$[\![\ c \gg\!\!=\![E \longrightarrow d], (d \gg\!\!=\![E \longrightarrow d])_{Info}\ ]\!] \cong ([\![\ c, \mathsf{id}\ ]\!]) \gg\!\!=_{Fam} (e \mapsto ((E\ e) \longrightarrow_{Fam} [\![\ d, \mathsf{id}\ ]\!])) \quad \square$$

Composition for UF

$$\_ \bullet_{\mathsf{Uni}} \_ : \mathsf{Uni}\ D \to \mathsf{UF}\ C\ D \to \mathsf{Uni}\ C$$

$$(\_ \bullet_{\mathsf{Info}} \_) : (c : \mathsf{Uni}\ D) \to (R : \mathsf{UF}\ C\ D) \to \mathsf{Info}\ (c \bullet_{\mathsf{Uni}} R) \to \mathsf{Info}\ c$$

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$$\begin{aligned}
\iota_{\mathsf{UF}} \bullet_{\mathsf{Uni}} R &= \iota_{\mathsf{UF}} \\
(\sigma_{\mathsf{UF}}\ c\ A) \bullet_{\mathsf{Uni}} R &= \sigma_{\mathsf{UF}}\ (c \bullet_{\mathsf{Uni}} R)\ (A \circ (c \bullet_{\mathsf{Info}} R)) \\
(\delta_{\mathsf{UF}}\ c\ A) \bullet_{\mathsf{Uni}} (d, \beta) &= (c \bullet_{\mathsf{Uni}} (d, \beta)) \gg\!\!= [(A \circ (c \bullet_{\mathsf{Info}} (d, \beta))) \longrightarrow d]
\end{aligned}$$

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$$\iota_{\mathsf{UF}} \bullet_{\mathsf{Uni}} R = \iota_{\mathsf{UF}}$$
$$(\sigma_{\mathsf{UF}}\ c\ A) \bullet_{\mathsf{Uni}} R = \sigma_{\mathsf{UF}}\ (c \bullet_{\mathsf{Uni}} R)\ (A \circ (c \bullet_{\mathsf{Info}} R))$$
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Theorem.
$\llbracket\, (c, \alpha) \bullet d \,\rrbracket\, Z = \llbracket\, c \bullet_{Uni} d, \alpha \circ (c \bullet_{Info}\ d) \,\rrbracket\, Z \cong \llbracket\, (c, \alpha) \,\rrbracket (\llbracket\, d \,\rrbracket\, Z).$ □

How suitable are uniform codes?

Uniform codes (most likely) capture a smaller class of functors compared to DS.

However all inductive-recursive definitions "in the wild" are already uniform (because coproducts definable).

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However all inductive-recursive definitions “in the wild” are already uniform (because coproducts definable).

Conjecture: UF and DS have the same proof-theoretical strength.

Summary

Uniform codes UF and polynomial codes PN as new, alternative axiomatisations of inductive-recursive definitions.

$$\mathrm{UF} \hookrightarrow \mathrm{DS} \hookrightarrow \mathrm{PN}$$

Both UF and PN are closed under composition; DS probably is not.

Existence of initial algebras for UF unproblematic. For PN, need to adjust the DS model slightly (but not much).

Are there other, even more well-behaved axiomatisations?

Thank you!