

# Comprehensive parametric polymorphism

Fredrik Nordvall Forsberg  
University of Strathclyde, Glasgow  
[fredrik.nordvall-forsberg@strath.ac.uk](mailto:fredrik.nordvall-forsberg@strath.ac.uk)

LFCS Seminar, Edinburgh, 3 May 2016

## Joint work with...



Neil Ghani (Strathclyde)



Alex Simpson (Ljubljana)

## Parametric polymorphism [Strachey, 1967]

- A polymorphic program

$$t : \forall \alpha. A$$

is **parametric** if it applies the same uniform algorithm at all instantiations  $t[B]$  of its type parameter.

- Typical example:

$$\text{reverse} : \forall \alpha. \text{List } \alpha \rightarrow \text{List } \alpha$$

## Reynolds insight: relational parametricity [1983]

- Turn the **negative** statement “not distinguishing types” into the **positive** statement “preserves all relations”.

## Reynolds insight: relational parametricity [1983]

- Turn the **negative** statement “not distinguishing types” into the **positive** statement “preserves all relations”.
- A polymorphic program  $t : \forall \alpha. A$  is **relationally parametric** if for all relations  $R \subseteq B \times B'$ ,

$$(t[B], t[B']) \in \langle A \rangle(R)$$

where  $\langle A \rangle(R) \subseteq A(B) \times A(B')$  is the **relational interpretation** of the type  $A$ .

- E.g. `reverse` :  $\forall \alpha. \text{List } \alpha \rightarrow \text{List } \alpha$  is relationally parametric.

# Applications of relational parametricity

Relational parametricity enables:

- Reasoning about **abstract data types**.
- Correctness (universal properties) of **encodings of data types**.
- ‘Theorems for **free!**’ [Wadler, 1989].
- Concretely, a specific example: if  $t : \forall \alpha. \alpha \rightarrow \alpha$  then  $t = \Lambda \alpha. \lambda x. x$ .

Usually in the setting of Girard’s/Reynold’s  $\lambda 2$  (System F) — serves as a model type theory for (impredicative) polymorphism.

*What is the fundamental category-theoretic structure needed to model relational parametricity for  $\lambda 2$ ?*

What is the fundamental category-theoretic structure needed to model relational parametricity for  $\lambda 2$ ?

- Perhaps surprising that this question does not have an established answer.

## What is the fundamental category-theoretic structure needed to model relational parametricity for $\lambda 2$ ?

- Perhaps surprising that this question does not have an established answer.
- We know the fundamental structure needed for  $\lambda 2$  ( $\lambda 2$  fibrations [Seely, 1987]).

## What is the fundamental category-theoretic structure needed to model relational parametricity for $\lambda 2$ ?

- Perhaps surprising that this question does not have an established answer.
- We know the fundamental structure needed for  $\lambda 2$  ( $\lambda 2$  fibrations [Seely, 1987]).
- We also know the fundamental structures used for relational parametricity (reflexive graph categories [Robinson and Rosolini, 1994], parametricity graphs [Dunphy and Reddy, 2004]).

## What is the fundamental category-theoretic structure needed to model relational parametricity for $\lambda 2$ ?

- Perhaps surprising that this question does not have an established answer.
- We know the fundamental structure needed for  $\lambda 2$  ( $\lambda 2$  fibrations [Seely, 1987]).
- We also know the fundamental structures used for relational parametricity (reflexive graph categories [Robinson and Rosolini, 1994], parametricity graphs [Dunphy and Reddy, 2004]).
- So why not just combine the two?

## So why not just combine the two?

- When doing so, the expected consequences of parametricity are only derivable if the underlying category is **well-pointed**.
- **Recall:** A category  $\mathbb{C}$  is well-pointed when  $f = g : A \longrightarrow B$  in  $\mathbb{C}$  iff  $f \circ e = g \circ e : \mathbf{1} \longrightarrow B$  for all global elements  $e : \mathbf{1} \longrightarrow A$ .
- This rules out many interesting categories, e.g. functor categories.

## So why not just combine the two?

- When doing so, the expected consequences of parametricity are only derivable if the underlying category is **well-pointed**.
- **Recall:** A category  $\mathbb{C}$  is well-pointed when  $f = g : A \longrightarrow B$  in  $\mathbb{C}$  iff  $f \circ e = g \circ e : \mathbf{1} \longrightarrow B$  for all global elements  $e : \mathbf{1} \longrightarrow A$ .
- This rules out many interesting categories, e.g. functor categories.
- Existing solutions (e.g. Birkedal and Møgelberg [2005]) circumvent this by adding significant additional structure to models (enough to model the full logic of Plotkin and Abadi).
- We seek instead a minimal solution still based on the idea of directly combining models of  $\lambda 2$  with structure for relational parametricity.

## A minimal solution

- We achieve this in a perhaps unexpected way: we change the notion of model of  $\lambda_2$ .
- $\lambda_2$  fibrations satisfying Lawvere's comprehension property.

## A minimal solution

- We achieve this in a perhaps unexpected way: we change the notion of model of  $\lambda 2$ .
- $\lambda 2$  fibrations satisfying Lawvere's **comprehension** property.
- This allows us to combine such **comprehensive**  $\lambda 2$  fibrations with reflexive graph structure to model relational parametricity for  $\lambda 2$ .
- Validating expected consequences, also for non-well-pointed categories.
- Proof involves novel ingredients due to minimality of structure:
  - ▶ definability of **direct image relations**,
  - ▶ arguments **without** use of **equality relations**, and
  - ▶ only weak forms of graph relations available ('**pseudographs**').

# Outline

- 1 The type theory  $\lambda 2$
- 2 Modelling  $\lambda 2$  using (comprehensive)  $\lambda 2$  fibrations
- 3 Modelling relational parametricity using (comprehensive) parametricity graphs
- 4 Reasoning about parametricity using a type theory  $\lambda 2R$

Breton City of Ys; the Cornish Land of Lyonesse (impossibly located between Cornwall and the Scilly Isles); the French Île Verte; the Portuguese Ilha Verde; all are variants of this legend. But if what the Egyptian priests really told Solon was that the disaster took place in the Far West, and that the survivors moved 'beyond the Pillars of Hercules', Atlantis can be easily identified.

It is the country of the Atlantians, mentioned by Diodorus Siculus (see page 448) as the most civilised people living to the westward of Lake Tritonis, from whom the Libyan Amazons, meaning the matriarchal tribes later described by Herodotus, seized their city of Cerne. Diodorus's legend cannot be archaeologically dated, but he makes it precede a Libyan invasion of the Aegean Islands and Thrace, an event which cannot have taken place later than the third millennium B.C. If, then, Atlantis was Western Libya, the floods which caused it to disappear may have been due either to a phenomenal rainfall such as caused the famous Mesopotamian and Ogygian Floods (see pages 138-9), or to a high tide with a strong north-westerly gale, such as washed away a large part of the Netherlands in the twelfth and thirteenth centuries and formed the Zuider Zee,\* or to a subsidence of the coast. Atlantis may, in fact, have been swamped at the formation of Lake Tritonis (see page 70). The area which was once covered several thousand square miles in the lower Nile valley, perhaps extended northward into the Western Mediterranean, and the geographer Scylax 'the Gulf of Tritonis', where the dangerous currents wash a chain of islands of which only Jerba and the Berkennahs survive.

The island left in the centre of the lake mentioned by Diodorus (see page 447) was perhaps the Chaamba Bou Rouba in the Sahara. Diodorus seems to be referring to such a catastrophe when he writes in his account of the Amazons and Atlantians (iii. 55): 'And it is said that, as a result of earthquakes, the parts of Libya towards the ocean engulfed Lake Tritonis, making it disappear.' Since Lake Tritonis still existed in his day, what he had probably been told was that 'as a result of earthquakes in the Western Mediterranean the sea engulfed part of Libya and formed Lake Tritonis.' The Zuider Zee and the Gopiac Lake have now both been reclaimed; and Lake Tritonis, which, according to Scylax, still covered 900 square miles in Classical times, has shrunk to the salt-marshes of Chott Melghir and Chott el Jerid. If this was Atlantis, some of the dispossessed agriculturists were driven west to Morocco, others south across the Sahara, others east to Egypt and beyond, taking their story with them; a few remained by the lakeside. Plato's elephants may well have been found in this territory, though the mountainous coastline of Atlantis belongs to Crete, of which the sea-hating Egyptians knew only by hearsay.

The five pairs of Poseidon's twin sons who took the oath of allegiance to Atlas have been representatives at Pharos of Keftiu kingdoms allied to the Cretans. In the Mycenaean Age double-sovereignty was the rule: Sparta with

\* Since this was written, history has repeated itself disastrously.

Castor and Polydeuces, Messenia with Idas and Lynceus, Argos with Proetus and Acrisius, Tiryns with Heracles and Iphicles, Thebes with Eteocles and Polyneices. Greed and cruelty will have been displayed by the Sons of Poseidon only after the fall of Cnossus, when commercial integrity declined and the merchant turned pirate.

Prometheus's name, 'forethought', may originate in a Greek misunderstanding of the Sanskrit word *pramanika*, the swastika, or fire-drill, which he had supposedly invented, since Zeus Prometheus at Thuri was shown holding a fire-drill. Prometheus, the Indo-European folk-hero, became confused with the Carian hero Palamedes; the inventor or distributor of all civilised arts (under the goddess's inspiration); and with the Babylonian god Ea, who claimed to have created a splendid man from the blood of Kingo (a son of Cronus), while the Mother-goddess Aruru created an inferior man from clay. The brothers Pramanthu and Manthu, who occur in the *Bhagavata Purana*, a Sanskrit epic, may be prototypes of Prometheus and Epimetheus ('afterthought'), yet Hesiod's account of Prometheus, Epimetheus, and Pandora is a genuine myth (see last paragraph, end of story), but an anti-genealogy, a folk-epic which, though based on the story of Genesis, is not a translation of it. Prometheus ('giving') was the Earth-god, the god of the soil, and that the sea was elsewhere (Aristophanes, *Birds* 971; Philostratus, *Life of Apollonius of Tyana* vi. 30), whom the pessimistic Hesiod blames for man's mortality and all the ills which beset life, as well as for the frivolous and unseemly behaviour of wives. His story of the division of the bull is equally unmythical; a comic anecdote, invented to account for Prometheus's punishment, and for the anomaly of presenting the gods only with the thigh-bones and fat cut from the sacrificial beast. In Genesis the sanctity of the thigh-bones is explained by Jacob's lameness which an angel inflicted on him during a wrestling match. Pandora's jar (not box) originally contained winged souls.

Greek islanders still carry fire from one place to another in the pith of fennel, and Prometheus's enchainment on Mount Caucasus may be a legend picked up by the Hellenes as they migrated to Greece from the Caspian Sea: of a frost-giant, recumbent on the snow of the high peaks, and attended by a flock of vultures.

The Athenians were at pains to deny that their goddess took Prometheus as her lover, which suggests that he had been locally identified with Hephaestus, another fire-god and inventor, of whom the same story was told (see page 98) because he shared a temple with Athene on the Acropolis.

Menoetius ('ruined strength') is a sacred king of the oak cult; the name refers perhaps to his ritual maiming (see pages 48 and 170).

While the right-handed swastika is a symbol of the sun, the left-handed is a symbol of the moon. Among the Akan of West Africa, a people of Libyo-Berber ancestry (see *introduction, end*), it represents the Triple-goddess Nyame.

## The type theory Λ2

# The polymorphic lambda calculus $\lambda 2$ (System F) [Girard, 1972; Reynolds, 1974]

- Four judgements:

$\Gamma$ ctxt	$\Gamma$ is a context
$\Gamma \vdash A$ type	$A$ is a type in context $\Gamma$
$\Gamma \vdash t : A$	term $t$ has type $A$ in context $\Gamma$
$\Gamma \vdash t = s : A$	judgemental equality

- Types and terms generated by grammars

$A, B ::= \alpha \mid A \rightarrow B \mid \forall \alpha. A$	types
$t, s ::= x \mid \lambda x. t \mid t s \mid \Lambda \alpha. t \mid t[B]$	terms

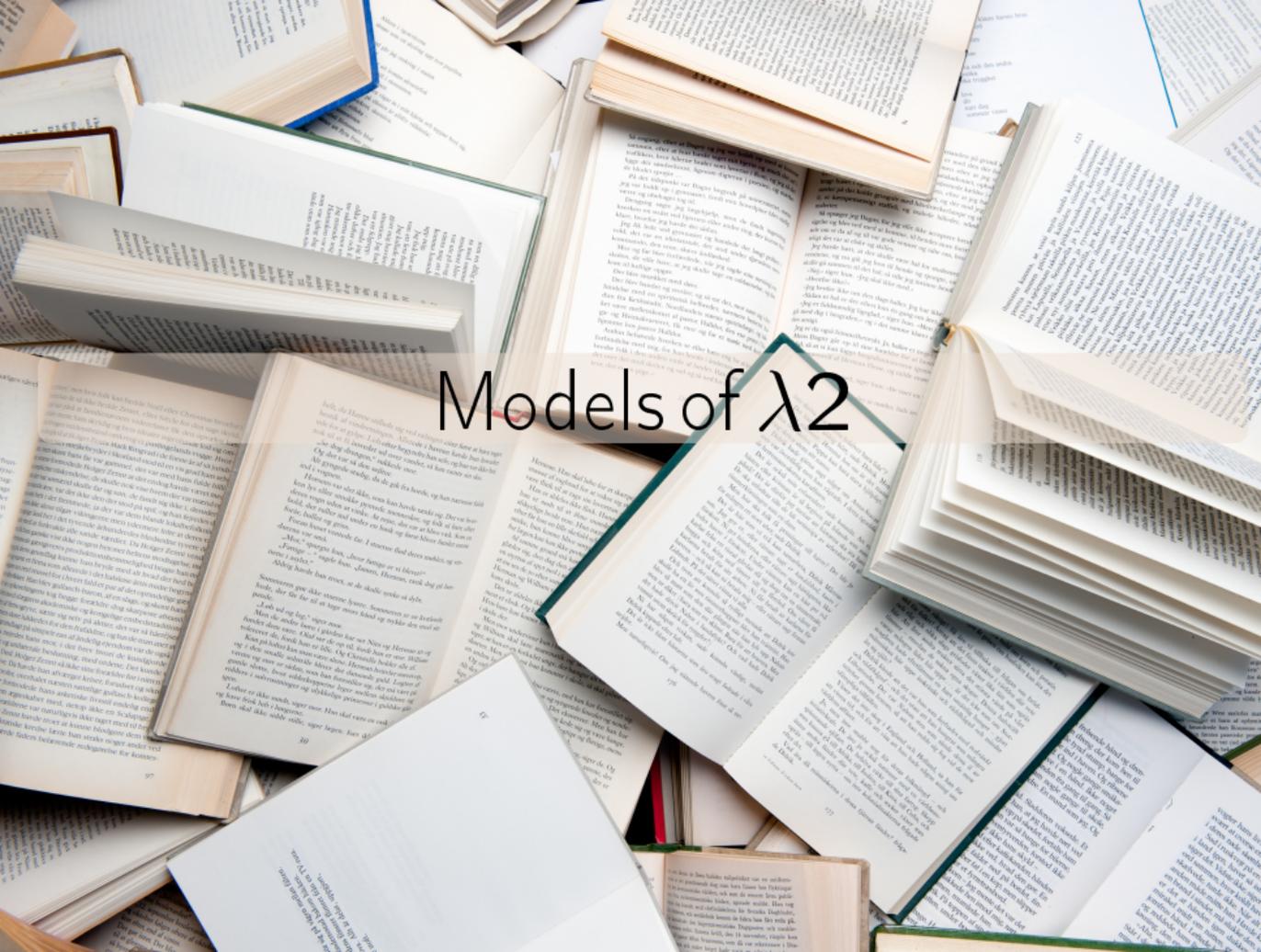
- Equality generated by  $(\beta)$  and  $(\eta)$  for both term and type abstraction.

## Only unusual feature of our presentation

- We use a **single** context with type and term variables interleaved.
- Standard from a dependent types perspective.
- Hence two different context extensions:

$$\frac{\Gamma \text{ ctxt}}{\Gamma, \alpha \text{ ctxt}} \quad (\alpha \notin \Gamma) \qquad \frac{\Gamma \text{ ctxt} \quad \Gamma \vdash A \text{ type}}{\Gamma, x : A \text{ ctxt}} \quad (x \notin \Gamma)$$

# Models of $\lambda 2$



## $\lambda 2$ fibrations [Seely, 1987; see also Jacobs, 1999]

### Definition ( $\lambda 2$ fibration)

A  $\lambda 2$  fibration is a fibration  $p : \mathbb{T} \rightarrow \mathbb{C}$ , where the base category  $\mathbb{C}$  has finite products, and the fibration:

- 1 is fibred cartesian closed;
- 2 has a generic object  $U$  — we write  $\Omega$  for  $p U$ ;
- 3 and has fibred-products along projections  $X \times \Omega \longrightarrow X$  in  $\mathbb{C}$ .

## $\lambda 2$ fibrations [Seely, 1987; see also Jacobs, 1999]

### Definition ( $\lambda 2$ fibration)

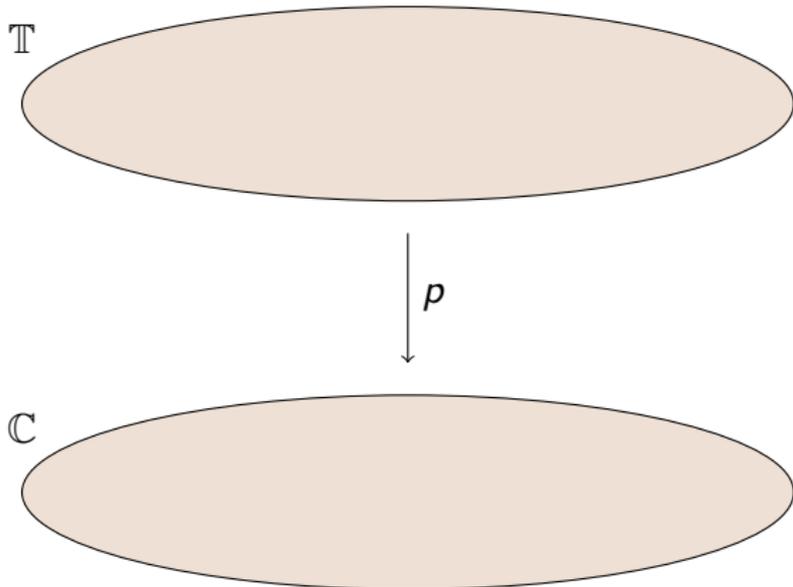
A  $\lambda 2$  fibration is a split fibration  $p : \mathbb{T} \rightarrow \mathbb{C}$ , where the base category  $\mathbb{C}$  has finite products, and the fibration:

- 1 is fibred cartesian closed;
- 2 has a split generic object  $U$  — we write  $\Omega$  for  $p U$ ;
- 3 and has fibred-products along projections  $X \times \Omega \longrightarrow X$  in  $\mathbb{C}$ .

Moreover, the reindexing functors given by the splitting should preserve the above-specified structure in fibres on the nose.

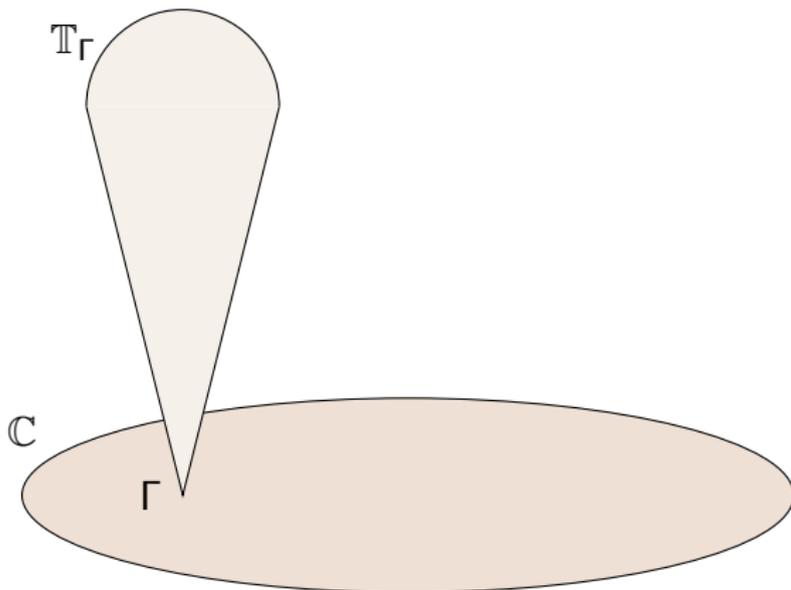
## Structure in detail (i)

- Fibration  $p : \mathbb{T} \rightarrow \mathbb{C}$ ,  $\mathbb{C}$  has finite products.
  - ▶  $\mathbb{C}$  category of type variable contexts and substitutions.
  - ▶ Products are context concatenation.



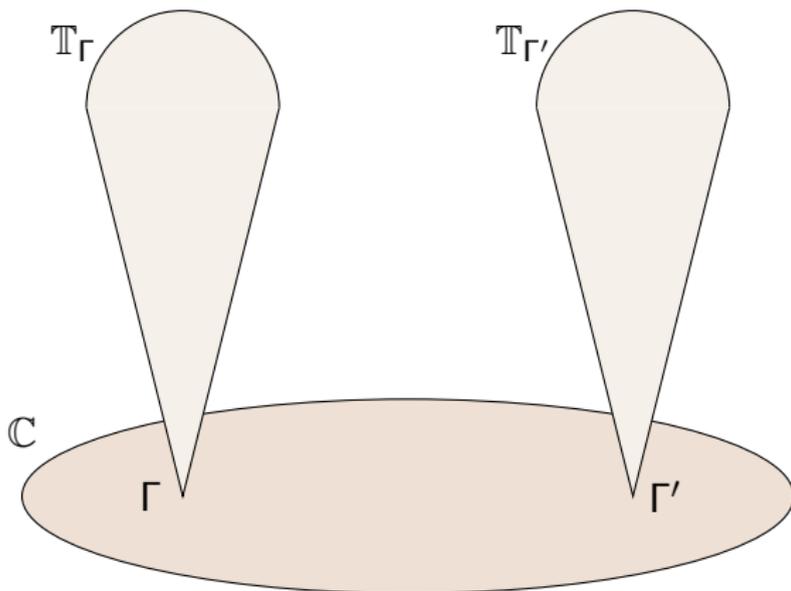
## Structure in detail (i)

- Fibration  $p : \mathbb{T} \rightarrow \mathbb{C}$ ,  $\mathbb{C}$  has finite products.
  - ▶  $\mathbb{C}$  category of type variable contexts and substitutions.
  - ▶ Products are context concatenation.
  - ▶ Fibre  $\mathbb{T}_\Gamma$  category of types in context  $\Gamma$ .



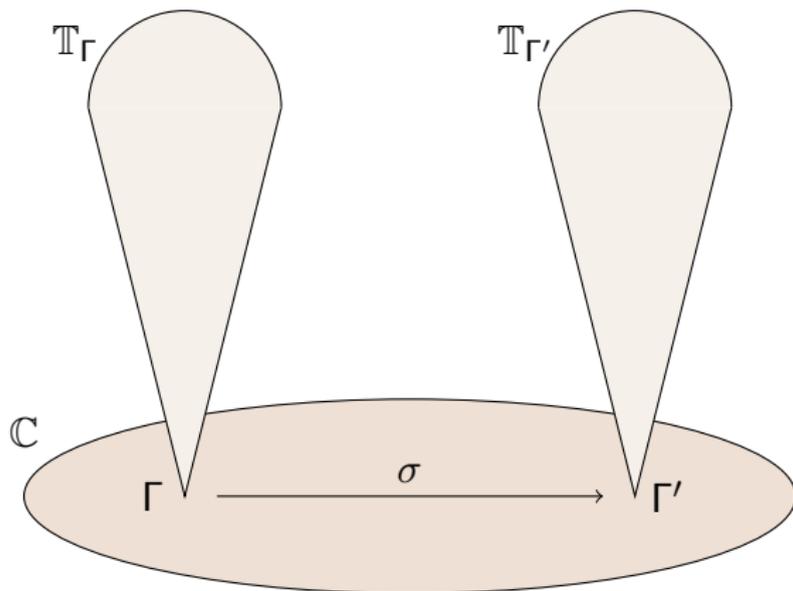
## Structure in detail (i)

- Fibration  $p : \mathbb{T} \rightarrow \mathbb{C}$ ,  $\mathbb{C}$  has finite products.
  - ▶  $\mathbb{C}$  category of type variable contexts and substitutions.
  - ▶ Products are context concatenation.
  - ▶ Fibre  $\mathbb{T}_\Gamma$  category of types in context  $\Gamma$ .



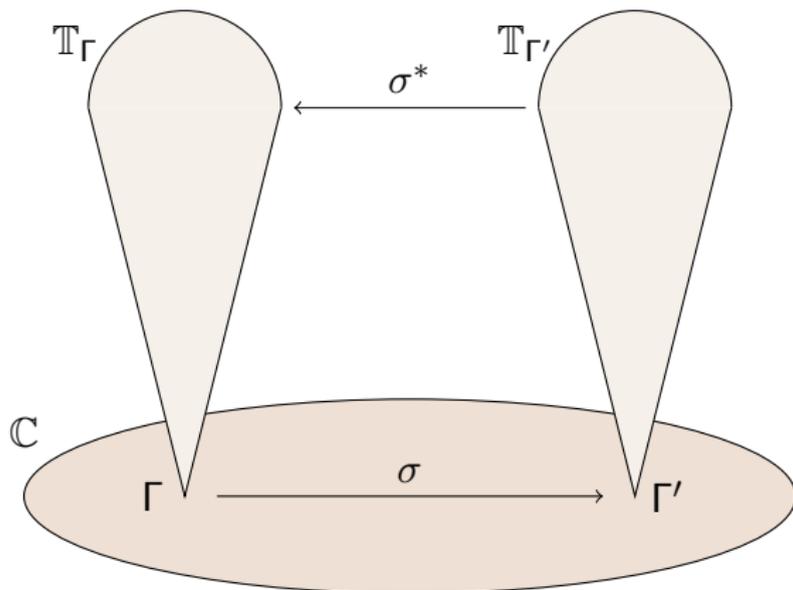
## Structure in detail (i)

- Fibration  $p : \mathbb{T} \rightarrow \mathbb{C}$ ,  $\mathbb{C}$  has finite products.
  - ▶  $\mathbb{C}$  category of type variable contexts and substitutions.
  - ▶ Products are context concatenation.
  - ▶ Fibre  $\mathbb{T}_\Gamma$  category of types in context  $\Gamma$ .



## Structure in detail (i)

- Fibration  $p : \mathbb{T} \rightarrow \mathbb{C}$ ,  $\mathbb{C}$  has finite products.
  - ▶  $\mathbb{C}$  category of type variable contexts and substitutions.
  - ▶ Products are context concatenation.
  - ▶ Fibre  $\mathbb{T}_\Gamma$  category of types in context  $\Gamma$ .
  - ▶ Reindexing is substitution.



## Structure in detail (ii)

- ... is fibred cartesian closed;
  - ▶ Each fibre is closed under exponentials.
  - ▶ Needed for  $\rightarrow$ .

## Structure in detail (ii)

- ... is fibred cartesian closed;
  - ▶ Each fibre is closed under exponentials.
  - ▶ Needed for  $\rightarrow$ .
- ... has a split generic object  $U$  — we write  $\Omega$  for  $p U$ ;
  - ▶ Every object  $A$  in  $\mathbb{T}$  arises as  $A \cong \sigma^*(U)$  for some  $\sigma : p(A) \longrightarrow \Omega$ .
  - ▶ “Every type arises uniquely by substitution from a generic type”.
  - ▶ The generic type  $U$  is a type variable  $\alpha$  in context  $\Omega = \alpha$ .
  - ▶ Needed for type variables.

## Structure in detail (ii)

- ... is fibred cartesian closed;
  - ▶ Each fibre is closed under exponentials.
  - ▶ Needed for  $\rightarrow$ .
- ... has a split generic object  $U$  — we write  $\Omega$  for  $p U$ ;
  - ▶ Every object  $A$  in  $\mathbb{T}$  arises as  $A \cong \sigma^*(U)$  for some  $\sigma : p(A) \rightarrow \Omega$ .
  - ▶ “Every type arises uniquely by substitution from a generic type”.
  - ▶ The generic type  $U$  is a type variable  $\alpha$  in context  $\Omega = \alpha$ .
  - ▶ Needed for type variables.
- ... and has fibred-products along projections  $\Gamma \times \Omega \rightarrow \Gamma$  in  $\mathbb{C}$ .
  - ▶ Each reindexing functor  $\pi_\Omega^* : \mathbb{T}_\Gamma \rightarrow \mathbb{T}_{\Gamma \times \Omega}$  has a right adjoint  $\prod_\Omega : \mathbb{T}_{\Gamma \times \Omega} \rightarrow \mathbb{T}_\Gamma$ .
  - ▶ Needed for  $\forall$ .

## Old-fashioned interpretation

- Given context  $\Gamma$ , let  $\Theta = \alpha_1, \dots, \alpha_n$  and  $\Delta = x_1 : A_1, \dots, x_m : A_m$  be the type and term variable components of  $\Gamma$ .

## Old-fashioned interpretation

- Given context  $\Gamma$ , let  $\Theta = \alpha_1, \dots, \alpha_n$  and  $\Delta = x_1 : A_1, \dots, x_m : A_m$  be the type and term variable components of  $\Gamma$ .
- Type variable context  $\Theta = \alpha_1, \dots, \alpha_n$  interpreted as  $\llbracket \Theta \rrbracket = \Omega^n$  in  $\mathbb{C}$ .

## Old-fashioned interpretation

- Given context  $\Gamma$ , let  $\Theta = \alpha_1, \dots, \alpha_n$  and  $\Delta = x_1 : A_1, \dots, x_m : A_m$  be the type and term variable components of  $\Gamma$ .
- Type variable context  $\Theta = \alpha_1, \dots, \alpha_n$  interpreted as  $[[\Theta]] = \Omega^n$  in  $\mathbb{C}$ .
- Type  $A$  in context  $\Theta$  is interpreted as an object in  $\mathbb{T}_{[[\Theta]]}$ .

## Old-fashioned interpretation

- Given context  $\Gamma$ , let  $\Theta = \alpha_1, \dots, \alpha_n$  and  $\Delta = x_1 : A_1, \dots, x_m : A_m$  be the type and term variable components of  $\Gamma$ .
- Type variable context  $\Theta = \alpha_1, \dots, \alpha_n$  interpreted as  $[[\Theta]] = \Omega^n$  in  $\mathbb{C}$ .
- Type  $A$  in context  $\Theta$  is interpreted as an object in  $\mathbb{T}_{[[\Theta]]}$ .
- Term variable context  $\Delta = x_1 : A_1, \dots, x_m : A_m$  interpreted as  $[[\Delta]] = [[A_1]] \times \dots \times [[A_m]]$  in  $\mathbb{T}_{[[\Theta]]}$ .

## Old-fashioned interpretation

- Given context  $\Gamma$ , let  $\Theta = \alpha_1, \dots, \alpha_n$  and  $\Delta = x_1 : A_1, \dots, x_m : A_m$  be the type and term variable components of  $\Gamma$ .
- Type variable context  $\Theta = \alpha_1, \dots, \alpha_n$  interpreted as  $\llbracket \Theta \rrbracket = \Omega^n$  in  $\mathbb{C}$ .
- Type  $A$  in context  $\Theta$  is interpreted as an object in  $\mathbb{T}_{\llbracket \Theta \rrbracket}$ .
- Term variable context  $\Delta = x_1 : A_1, \dots, x_m : A_m$  interpreted as  $\llbracket \Delta \rrbracket = \llbracket A_1 \rrbracket \times \dots \times \llbracket A_m \rrbracket$  in  $\mathbb{T}_{\llbracket \Theta \rrbracket}$ .
- Term  $\Gamma \vdash t : A$  is interpreted as morphism

$$\llbracket t \rrbracket_{\Theta; \Delta} : \llbracket \Delta \rrbracket \rightarrow \llbracket A \rrbracket \quad \text{in } \mathbb{T}_{\llbracket \Theta \rrbracket}$$

## Old-fashioned interpretation

- Given context  $\Gamma$ , let  $\Theta = \alpha_1, \dots, \alpha_n$  and  $\Delta = x_1 : A_1, \dots, x_m : A_m$  be the type and term variable components of  $\Gamma$ .
- Type variable context  $\Theta = \alpha_1, \dots, \alpha_n$  interpreted as  $\llbracket \Theta \rrbracket = \Omega^n$  in  $\mathbb{C}$ .
- Type  $A$  in context  $\Theta$  is interpreted as an object in  $\mathbb{T}_{\llbracket \Theta \rrbracket}$ .
- Term variable context  $\Delta = x_1 : A_1, \dots, x_m : A_m$  interpreted as  $\llbracket \Delta \rrbracket = \llbracket A_1 \rrbracket \times \dots \times \llbracket A_m \rrbracket$  in  $\mathbb{T}_{\llbracket \Theta \rrbracket}$ .
- Term  $\Gamma \vdash t : A$  is interpreted as morphism

$$\llbracket t \rrbracket_{\Theta; \Delta} : \llbracket \Delta \rrbracket \rightarrow \llbracket A \rrbracket \quad \text{in } \mathbb{T}_{\llbracket \Theta \rrbracket}$$

- The combined context made things awkward; let's fix that by modifying the notion of model and giving a new interpretation.

## Our modification: one new ingredient

We take inspirations from models of dependent types, where separated contexts are not possible.

### Definition (Comprehensive $\lambda 2$ fibration)

A  $\lambda 2$  fibration  $p : \mathbb{T} \rightarrow \mathbb{C}$  is *comprehensive* if it enjoys the *comprehension property*: the fibred-terminal-object functor  $X \mapsto \mathbf{1}_X : \mathbb{C} \rightarrow \mathbb{T}$  has a specified right adjoint  $K : \mathbb{T} \rightarrow \mathbb{C}$ .

- Given  $A \in \mathbb{T}_\Gamma$ , think of  $K(A)$  as the extended context  $\Gamma, x : A$ .
- For  $A \in \mathbb{T}_\Gamma$ , write  $\kappa_A = p(\varepsilon_A) : K(A) \longrightarrow \Gamma$  for the ‘projection’ map obtained by applying  $p$  to the counit  $\varepsilon_A : \mathbf{1}_{K(A)} \longrightarrow A$  in  $\mathbb{T}$ .

## Interpretation in a comprehensive $\lambda 2$ fibration

- Contexts  $\Gamma$  interpreted as object  $\llbracket \Gamma \rrbracket$  in  $\mathbb{C}$ .
- Type  $\Gamma \vdash A$  type interpreted as object  $\llbracket A \rrbracket_{\Gamma}$  in  $\mathbb{T}_{\llbracket \Gamma \rrbracket}$ .

## Interpretation in a comprehensive $\lambda 2$ fibration

- Contexts  $\Gamma$  interpreted as object  $\llbracket \Gamma \rrbracket$  in  $\mathbb{C}$ .
- Type  $\Gamma \vdash A$  type interpreted as object  $\llbracket A \rrbracket_\Gamma$  in  $\mathbb{T}_{\llbracket \Gamma \rrbracket}$ .
- Mutually defined, simultaneously with maps  $\pi_\Gamma^\alpha : \llbracket \Gamma \rrbracket \longrightarrow \Omega$  for every context  $\Gamma$  containing  $\alpha$ .

$$\llbracket \cdot \rrbracket = \mathbf{1}$$

$$\llbracket \alpha \rrbracket_\Gamma = (\pi_\Gamma^\alpha)^* U$$

$$\llbracket \Gamma, \alpha \rrbracket = \llbracket \Gamma \rrbracket \times \Omega$$

$$\llbracket A \rightarrow B \rrbracket_\Gamma = \llbracket A \rrbracket_\Gamma \Rightarrow_{\llbracket \Gamma \rrbracket} \llbracket B \rrbracket_\Gamma$$

$$\llbracket \Gamma, x : A \rrbracket = K \llbracket A \rrbracket_\Gamma$$

$$\llbracket \forall \alpha. A \rrbracket = \prod_{\Omega} \llbracket A \rrbracket_{\Gamma, \alpha}$$

$$\pi_{\Gamma, \alpha}^\alpha = \pi_2$$

$$\pi_{\Gamma, \beta}^\alpha = \pi_\Gamma^\alpha \circ \pi_1 \quad (\beta \neq \alpha)$$

$$\pi_{\Gamma, x:A}^\alpha = \pi_\Gamma^\alpha \circ \kappa_{\llbracket A \rrbracket_\Gamma}$$

## Interpretation in a comprehensive $\lambda 2$ fibration

- Contexts  $\Gamma$  interpreted as object  $\llbracket \Gamma \rrbracket$  in  $\mathbb{C}$ .
- Type  $\Gamma \vdash A$  type interpreted as object  $\llbracket A \rrbracket_\Gamma$  in  $\mathbb{T}_{\llbracket \Gamma \rrbracket}$ .
- Mutually defined, simultaneously with maps  $\pi_\Gamma^\alpha : \llbracket \Gamma \rrbracket \longrightarrow \Omega$  for every context  $\Gamma$  containing  $\alpha$ .

$$\begin{aligned} \llbracket \cdot \rrbracket &= \mathbf{1} & \llbracket \alpha \rrbracket_\Gamma &= (\pi_\Gamma^\alpha)^* U \\ \llbracket \Gamma, \alpha \rrbracket &= \llbracket \Gamma \rrbracket \times \Omega & \llbracket A \rightarrow B \rrbracket_\Gamma &= \llbracket A \rrbracket_\Gamma \Rightarrow_{\llbracket \Gamma \rrbracket} \llbracket B \rrbracket_\Gamma \\ \llbracket \Gamma, x : A \rrbracket &= K \llbracket A \rrbracket_\Gamma & \llbracket \forall \alpha. A \rrbracket &= \prod_{\Omega} \llbracket A \rrbracket_{\Gamma, \alpha} \end{aligned}$$

$$\pi_{\Gamma, \alpha}^\alpha = \pi_2 \quad \pi_{\Gamma, \beta}^\alpha = \pi_\Gamma^\alpha \circ \pi_1 \quad (\beta \neq \alpha) \quad \pi_{\Gamma, x:A}^\alpha = \pi_\Gamma^\alpha \circ \kappa_{\llbracket A \rrbracket_\Gamma}$$

- Term  $\Gamma \vdash t : A$  is interpreted as global element

$$\llbracket t \rrbracket_\Gamma : \mathbf{1}_{\llbracket \Gamma \rrbracket} \longrightarrow \llbracket A \rrbracket_\Gamma \quad \text{in } \mathbb{T}_{\llbracket \Gamma \rrbracket}$$

## For future reference

Compare the interpretation of terms in standard and comprehensive  $\lambda 2$  fibrations:

- $\llbracket t \rrbracket_{\Theta; \Delta} : \llbracket \Delta \rrbracket \rightarrow \llbracket A \rrbracket$  in  $\mathbb{T}_{\llbracket \Theta \rrbracket}$  (old-fashioned, standard)
- versus global element

$$\llbracket t \rrbracket_{\Gamma} : \mathbf{1}_{\llbracket \Gamma \rrbracket} \longrightarrow \llbracket A \rrbracket_{\Gamma} \quad \text{in } \mathbb{T}_{\llbracket \Gamma \rrbracket}$$

(comprehensive)

# Soundness and completeness

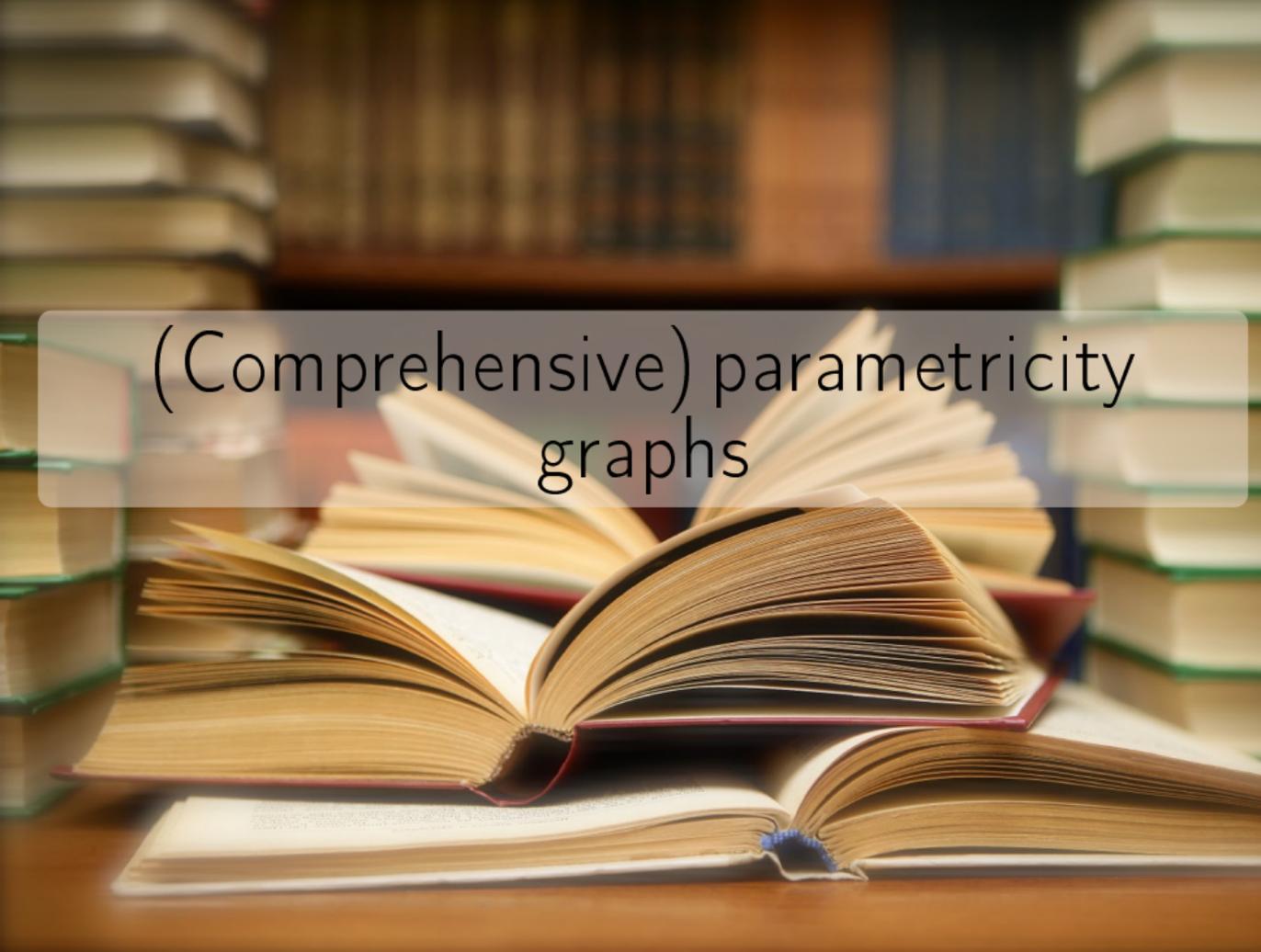
## Theorem (Soundness for $\lambda 2$ )

If  $\Gamma \vdash t_1 = t_2 : A$  then, in every comprehensive  $\lambda 2$  fibration, we have  $\llbracket t_1 \rrbracket_\Gamma = \llbracket t_2 \rrbracket_\Gamma$ .

## Theorem (Full completeness for $\lambda 2$ )

There exists a comprehensive  $\lambda 2$  fibration satisfying:

- 1 for every type  $\Gamma \vdash A$  type, every global point  $\mathbf{1}_{\llbracket \Gamma \rrbracket} \longrightarrow \llbracket A \rrbracket_\Gamma$  is the denotation  $\llbracket t \rrbracket_\Gamma$  of some term  $\Gamma \vdash t : A$ ; and
- 2 for all terms  $\Gamma \vdash t_1, t_2 : A$  satisfying  $\llbracket t_1 \rrbracket_\Gamma = \llbracket t_2 \rrbracket_\Gamma$ , we have  $\Gamma \vdash t_1 = t_2 : A$ .

The background of the slide is a photograph of a library or study area. In the foreground, several books are open, showing their pages. The books are arranged in a way that they appear to be part of a collection. The lighting is warm, highlighting the texture of the paper and the spines of the books. A semi-transparent white box is overlaid on the image, containing the text.

(Comprehensive) parametricity  
graphs

## Incorporating relational parametricity

- These models do not model parametricity.
- In order to do so, we combine with the structure of reflexive graph categories [Ma and Reynolds, 1992; Robinson and Rosolini, 1994; O'Hearn and Tennent, 1995; ...].
- Simple category-theoretic structure for modelling relations.

## Reflexive graph categories

$$\mathbb{E} \begin{array}{c} \xrightarrow{\nabla_1} \\ \xleftarrow{\Delta} \\ \xrightarrow{\nabla_2} \end{array} \mathbb{V}$$

- Categories  $\mathbb{V}$  and  $\mathbb{E}$ , where we think of  $\mathbb{E}$  as category of **relations** over objects of  $\mathbb{V}$ .
- The functors  $\nabla_1, \nabla_2$  are 'projection' functors giving **source** and **target** of relations, respectively, and  $\Delta$  maps an object to its '**identity relation**'.

## Reflexive graph categories

$$\begin{array}{ccc} & \nabla_1 & \\ \mathbb{E} & \xrightarrow{\quad} & \mathbb{V} \\ & \Delta & \\ & \nabla_2 & \end{array}$$

- Categories  $\mathbb{V}$  and  $\mathbb{E}$ , where we think of  $\mathbb{E}$  as category of **relations** over objects of  $\mathbb{V}$ .
- The functors  $\nabla_1, \nabla_2$  are 'projection' functors giving **source** and **target** of relations, respectively, and  $\Delta$  maps an object to its '**identity relation**'.
- Notation:  $R: A \leftrightarrow B$  means  $R \in \mathbb{E}$  and  $\nabla_1 R = A, \nabla_2 R = B$ .
- Similarly, write  $f \times g: R \longrightarrow S$  if there is  $h: R \longrightarrow S$  in  $\mathbb{E}$  with  $\nabla_1 h = f$  and  $\nabla_2 h = g$ . (Will soon assume  $h$  is unique, if it exists.)

## Parametricity graphs [Dunphy, 2002; Dunphy and Reddy, 2004]

$$\mathbb{E} \begin{array}{c} \xrightarrow{\nabla_1} \\ \xleftarrow{\Delta} \\ \xrightarrow{\nabla_2} \end{array} \mathbb{V}$$

- We need to add further conditions to ensure that the objects of  $\mathbb{E}$  behave sufficiently like relations.

## Parametricity graphs [Dunphy, 2002; Dunphy and Reddy, 2004]

$$\mathbb{E} \begin{array}{c} \xrightarrow{\nabla_1} \\ \xleftarrow{\Delta} \\ \xrightarrow{\nabla_2} \end{array} \mathbb{V}$$

- We need to add further conditions to ensure that the objects of  $\mathbb{E}$  behave sufficiently like relations.
- **Relational** if  $\langle \nabla_1, \nabla_2 \rangle: \mathbb{E} \rightarrow \mathbb{V} \times \mathbb{V}$  is faithful. Intuitively, relations are **proof-irrelevant**.

## Parametricity graphs [Dunphy, 2002; Dunphy and Reddy, 2004]

$$\begin{array}{ccc} & \nabla_1 & \\ \mathbb{E} & \xrightarrow{\quad} & \mathbb{V} \\ & \Delta & \\ & \xleftarrow{\quad} & \\ & \nabla_2 & \end{array}$$

- We need to add further conditions to ensure that the objects of  $\mathbb{E}$  behave sufficiently like relations.
- **Relational** if  $\langle \nabla_1, \nabla_2 \rangle: \mathbb{E} \rightarrow \mathbb{V} \times \mathbb{V}$  is faithful. Intuitively, relations are **proof-irrelevant**.
- **Identity property** if for every  $h: \Delta A \longrightarrow \Delta B$  in  $\mathbb{E}$ , it holds that  $\nabla_1 h = \nabla_2 h$ . Allows one to think of  $\Delta A$  as an **identity relation** on  $A$ .

## Parametricity graphs [Dunphy, 2002; Dunphy and Reddy, 2004]

$$\begin{array}{ccc} & \nabla_1 & \\ \mathbb{E} & \xrightarrow{\quad} & \mathbb{V} \\ & \Delta & \\ & \nabla_2 & \end{array}$$

- We need to add further conditions to ensure that the objects of  $\mathbb{E}$  behave sufficiently like relations.
- **Relational** if  $\langle \nabla_1, \nabla_2 \rangle: \mathbb{E} \rightarrow \mathbb{V} \times \mathbb{V}$  is faithful. Intuitively, relations are **proof-irrelevant**.
- **Identity property** if for every  $h: \Delta A \longrightarrow \Delta B$  in  $\mathbb{E}$ , it holds that  $\nabla_1 h = \nabla_2 h$ . Allows one to think of  $\Delta A$  as an **identity relation** on  $A$ .
- **Parametricity graph**: relational, with the identity property, and  $\langle \nabla_1, \nabla_2 \rangle: \mathbb{E} \rightarrow \mathbb{V} \times \mathbb{V}$  a fibration. Ensures that there are enough relations by supplying **inverse image relations**.

# Combining reflexive graphs and comprehensive $\lambda 2$ fibrations

# Combining reflexive graphs and comprehensive $\lambda 2$ fibrations

## Main definition (Comprehensive $\lambda 2$ parametricity graph)

A **comprehensive  $\lambda 2$  parametricity graph** is a reflexive graph of comprehensive  $\lambda 2$  fibrations

$$\begin{array}{ccc} \mathcal{R}(\mathbb{T}) & \begin{array}{c} \xrightarrow{\nabla_1^{\mathbb{T}}, \Delta^{\mathbb{T}}, \nabla_2^{\mathbb{T}}} \\ \xleftarrow{\hspace{1.5cm}} \\ \xrightarrow{\hspace{1.5cm}} \\ \xleftarrow{\hspace{1.5cm}} \end{array} & \mathbb{T} \\ \downarrow p^{\mathcal{R}} & & \downarrow p \\ \mathcal{R}(\mathbb{C}) & \begin{array}{c} \xrightarrow{\hspace{1.5cm}} \\ \xleftarrow{\hspace{1.5cm}} \\ \xrightarrow{\hspace{1.5cm}} \\ \xleftarrow{\hspace{1.5cm}} \end{array} & \mathbb{C} \\ & \nabla_1^{\mathbb{C}}, \Delta^{\mathbb{C}}, \nabla_2^{\mathbb{C}} & \end{array}$$

which is “fibrewise” a parametricity graph.

# Combining reflexive graphs and comprehensive $\lambda 2$ fibrations

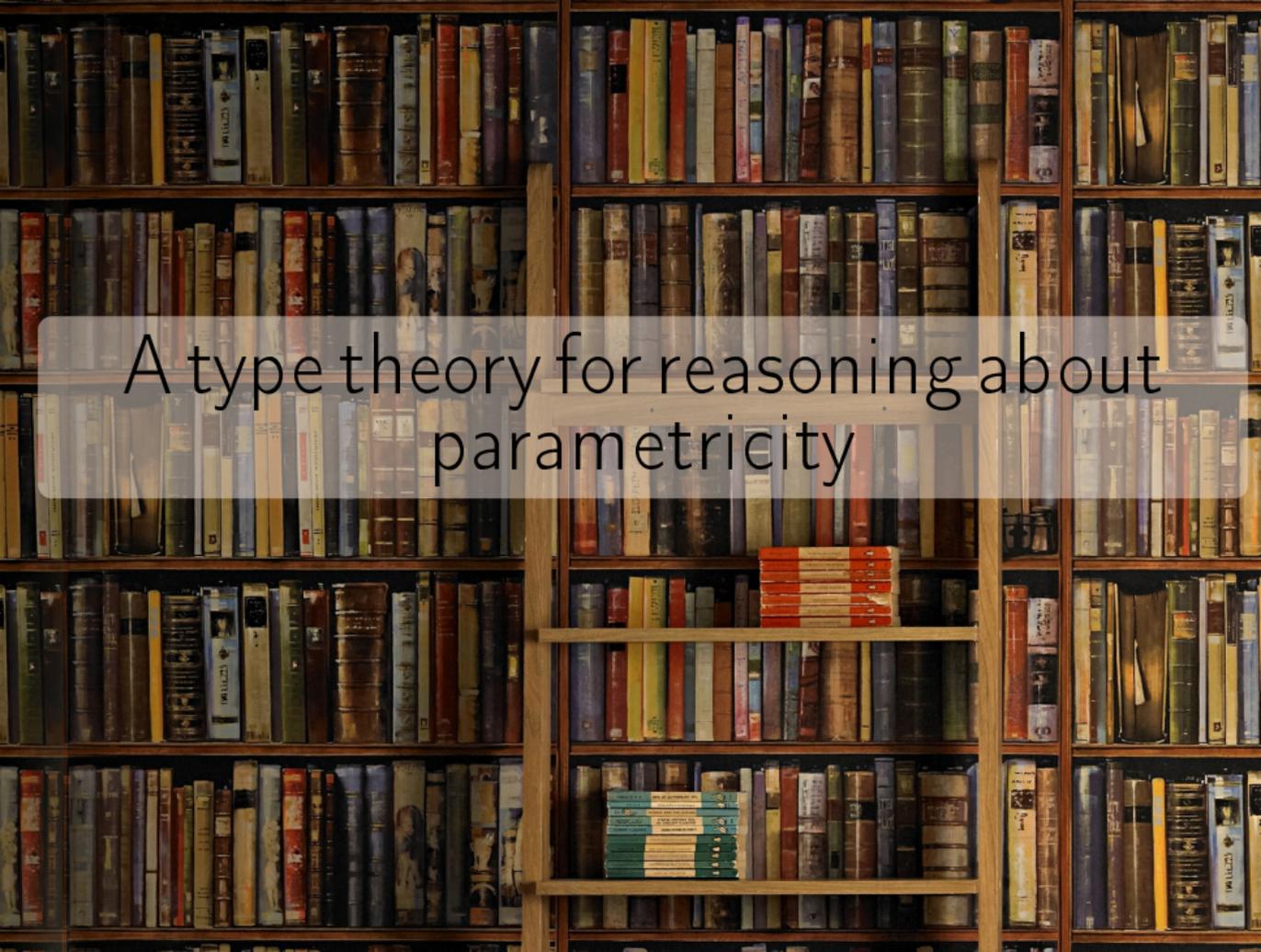
## Main definition (Comprehensive $\lambda 2$ parametricity graph)

A **comprehensive  $\lambda 2$  parametricity graph** is a reflexive graph of comprehensive  $\lambda 2$  fibrations

$$\begin{array}{ccc} \mathcal{R}(\mathbb{T}) & \begin{array}{c} \xrightarrow{\nabla_1^{\mathbb{T}}, \Delta^{\mathbb{T}}, \nabla_2^{\mathbb{T}}} \\ \xleftarrow{\hspace{1.5cm}} \\ \xrightarrow{\hspace{1.5cm}} \\ \xleftarrow{\hspace{1.5cm}} \end{array} & \mathbb{T} \\ \downarrow p^{\mathcal{R}} & & \downarrow p \\ \mathcal{R}(\mathbb{C}) & \begin{array}{c} \xrightarrow{\hspace{1.5cm}} \\ \xleftarrow{\hspace{1.5cm}} \\ \xrightarrow{\hspace{1.5cm}} \\ \xleftarrow{\hspace{1.5cm}} \end{array} & \mathbb{C} \\ & \nabla_1^{\mathbb{C}}, \Delta^{\mathbb{C}}, \nabla_2^{\mathbb{C}} & \end{array}$$

which is “fibrewise” a parametricity graph.

**Note:** Recover “broken” definition by dropping **comprehensive**.



A type theory for reasoning about  
parametricity

## Reasoning in models: a type theory $\lambda 2R$

- We construct a type theory  $\lambda 2R$  which is the 'internal language' of comprehensive  $\lambda 2$  parametricity graphs.
- By proving soundness and completeness, we can work in  $\lambda 2R$  instead of directly in the model.
- $\lambda 2R$  is similar in many respects to System R [Abadi, Cardelli and Curien, 1993] and System P [Dunphy, 2002].

## Reasoning in models: a type theory $\lambda 2R$

- We construct a type theory  $\lambda 2R$  which is the ‘internal language’ of comprehensive  $\lambda 2$  parametricity graphs.
- By proving soundness and completeness, we can work in  $\lambda 2R$  instead of directly in the model.
- $\lambda 2R$  is similar in many respects to System R [Abadi, Cardelli and Curien, 1993] and System P [Dunphy, 2002].
- *Not* a conservative extension of  $\lambda 2$  — parametric models enjoy much stronger properties than arbitrary models (for which  $\lambda 2$  is internal language).

## New judgement forms

$\lambda 2R$  extends  $\lambda 2$  with three new judgements:

$\Theta$  rctxt       $\Theta$  is a relational context

$\Theta \vdash A_1 R A_2$  rel       $R$  is a relation between types  $A_1$  and  $A_2$

$\Theta \vdash (t_1 : A_1) R (t_2 : A_2)$        $t_1 : A_1$  is related to  $t_2 : A_2$  by the relation  $R$

## Relation formation rules

$$\frac{}{\Theta \vdash \alpha\rho\beta \text{ rel}} \quad (\alpha\rho\beta \in \Theta)$$

$$\frac{\Theta \vdash A_1 R A_2 \text{ rel} \quad \Theta \vdash B_1 S B_2 \text{ rel}}{\Theta \vdash (A_1 \rightarrow B_1)(R \rightarrow S)(A_2 \rightarrow B_2) \text{ rel}}$$

$$\frac{\Theta, \alpha\rho\beta \vdash A_1 R A_2 \text{ rel}}{\Theta \vdash (\forall\alpha. A_1)(\forall\alpha\rho\beta. R)(\forall\beta. A_2) \text{ rel}}$$

## Relation formation rules

$$\frac{}{\Theta \vdash \alpha\rho\beta \text{ rel}} \quad (\alpha\rho\beta \in \Theta) \qquad \frac{\Theta \vdash A_1 R A_2 \text{ rel} \quad \Theta \vdash B_1 S B_2 \text{ rel}}{\Theta \vdash (A_1 \rightarrow B_1)(R \rightarrow S)(A_2 \rightarrow B_2) \text{ rel}}$$

$$\frac{\Theta, \alpha\rho\beta \vdash A_1 R A_2 \text{ rel}}{\Theta \vdash (\forall\alpha. A_1)(\forall\alpha\rho\beta. R)(\forall\beta. A_2) \text{ rel}}$$

$$\frac{\Theta \vdash B_1 R B_2 \text{ rel} \quad (\Theta)_1 \vdash t_1 : A_1 \rightarrow B_1 \quad (\Theta)_2 \vdash t_2 : A_2 \rightarrow B_2}{\Theta \vdash A_1([t_1 \times t_2]^{-1} R) A_2 \text{ rel}}$$

(Will get back to projections  $(-)_i$  soon.)

## Direct image relations

Direct image relations

$$\frac{\Theta \vdash A_1 R A_2 \text{ rel} \quad (\Theta)_1 \vdash t_1 : A_1 \rightarrow B_1 \quad (\Theta)_2 \vdash t_2 : A_2 \rightarrow B_2}{\Theta \vdash B_1([t_1 \times t_2]!R)B_2 \text{ rel}}$$

## Direct image relations

Direct image relations

$$\frac{\Theta \vdash A_1 R A_2 \text{ rel} \quad (\Theta)_1 \vdash t_1 : A_1 \rightarrow B_1 \quad (\Theta)_2 \vdash t_2 : A_2 \rightarrow B_2}{\Theta \vdash B_1([t_1 \times t_2]!R)B_2 \text{ rel}}$$

are **definable** by the impredicative encoding

$$[t_1 \times t_2]!R := [i_{B_1} \times i_{B_2}]^{-1}(\forall \alpha \rho \beta. ([(- \circ t_1) \times (- \circ t_2)]^{-1}(R \rightarrow \rho)) \rightarrow \rho)$$

where  $i_B$  abbreviates  $\lambda b. \Lambda \alpha. \lambda t. t b : B \rightarrow \forall \alpha. (B \rightarrow \alpha) \rightarrow \alpha$ .

## Direct image relations

Direct image relations

$$\frac{\Theta \vdash A_1 R A_2 \text{ rel} \quad (\Theta)_1 \vdash t_1 : A_1 \rightarrow B_1 \quad (\Theta)_2 \vdash t_2 : A_2 \rightarrow B_2}{\Theta \vdash B_1([t_1 \times t_2]!R)B_2 \text{ rel}}$$

are **definable** by an impredicative encoding.

## Direct image relations

Direct image relations

$$\frac{\Theta \vdash A_1 R A_2 \text{ rel} \quad (\Theta)_1 \vdash t_1 : A_1 \rightarrow B_1 \quad (\Theta)_2 \vdash t_2 : A_2 \rightarrow B_2}{\Theta \vdash B_1([t_1 \times t_2]!R)B_2 \text{ rel}}$$

are **definable** by an impredicative encoding.

Semantically, this means:

### Theorem

*In any comprehensive  $\lambda 2$  parametricity graph, the functors*

$$\langle \nabla_1^{\mathbb{T}}, \nabla_2^{\mathbb{T}} \rangle \downarrow_{\mathcal{R}(\mathbb{T})_W} : \mathcal{R}(\mathbb{T})_W \rightarrow \mathbb{T}_{\nabla_1^{\mathbb{C}}W} \times \mathbb{T}_{\nabla_2^{\mathbb{C}}W}$$

*are also **opfibrations** (hence bifibrations).*

## Operations on syntax

- Left and right **projections**  $(\cdot)_1, (\cdot)_2$  from relational contexts to typing contexts.

$$(\cdot)_i = \cdot$$

$$(\Theta, \alpha_1 \rho \alpha_2)_i = (\Theta)_i, \alpha_i$$

$$(\Theta, (x_1 : A_1)R(x_2 : A_2))_i = (\Theta)_i, x_i : A_i$$

## Operations on syntax

- Left and right **projections**  $(\cdot)_1$ ,  $(\cdot)_2$  from relational contexts to typing contexts.

$$(\cdot)_i = \cdot$$

$$(\Theta, \alpha_1 \rho \alpha_2)_i = (\Theta)_i, \alpha_i$$

$$(\Theta, (x_1 : A_1)R(x_2 : A_2))_i = (\Theta)_i, x_i : A_i$$

- Conversely, a “**doubling**” operation takes typing contexts to relational contexts.
- Mutually defined with a “**relational interpretation**”  $\langle A \rangle$  of types  $A$ .

$$\langle \cdot \rangle = \cdot$$

$$\langle \alpha \rangle = \rho^\alpha$$

$$\langle \Gamma, \alpha \rangle = \langle \Gamma \rangle, \alpha \rho^\alpha \alpha$$

$$\langle A \rightarrow B \rangle = \langle A \rangle \rightarrow \langle B \rangle$$

$$\langle \Gamma, x : A \rangle = \langle \Gamma \rangle, (x : A) \langle A \rangle (x : A)$$

$$\langle \forall \alpha. A \rangle = \forall \alpha \rho^\alpha \alpha. \langle A \rangle$$

## Operations on syntax

- Left and right **projections**  $(\cdot)_1, (\cdot)_2$  from relational contexts to typing contexts.

$$\begin{aligned}(\cdot)_i &= \cdot \\(\Theta, \alpha_1 \rho \alpha_2)_i &= (\Theta)_i, \alpha_i \\(\Theta, (x_1 : A_1)R(x_2 : A_2))_i &= (\Theta)_i, x_i : A_i\end{aligned}$$

- Conversely, a “**doubling**” operation takes typing contexts to relational contexts.
- Mutually defined with a “**relational interpretation**”  $\langle \cdot \rangle$  of types  $A$ .

$$\begin{aligned}\langle \cdot \rangle &= \cdot & \langle \alpha \rangle &= \rho^\alpha \\ \langle \Gamma, \alpha \rangle &= \langle \Gamma \rangle, \alpha \rho^\alpha \alpha & \langle A \rightarrow B \rangle &= \langle A \rangle \rightarrow \langle B \rangle \\ \langle \Gamma, x : A \rangle &= \langle \Gamma \rangle, (x : A) \langle A \rangle (x : A) & \langle \forall \alpha. A \rangle &= \forall \alpha \rho^\alpha \alpha. \langle A \rangle\end{aligned}$$

- Note:** Left and right hand side treated separately, so e.g.  $\alpha \rho^\alpha \alpha$  equivalent to  $\alpha \rho \beta$  if everything fresh.

## Reflexive graph structure on syntax

### Lemma

- 1 If  $\Theta \vdash (t_1 : A_1)R(t_2 : A_2)$  then  $(\Theta)_i \vdash t_i : A_i$ .
- 2 If  $\Gamma \vdash t : A$  then  $\langle \Gamma \rangle \vdash (t : A)\langle A \rangle(t : A)$ .

Second item is Reynolds' **Abstraction Theorem** in our setting.

## Relatedness rules: standard relation formers

$$\frac{}{\Theta \vdash (x_1 : A_1)R(x_2 : A_2)} \quad ((x_1 : A_1)R(x_2 : A_2) \in \Theta)$$

$$\frac{\Theta, (x_1 : A_1)R(x_2 : A_2) \vdash (t_1 : B_1)S(t_2 : B_2)}{\Theta \vdash (\lambda x_1. t_1 : A_1 \rightarrow B_1)(R \rightarrow S)(\lambda x_2. t_2 : A_2 \rightarrow B_2)}$$

$$\frac{\Theta \vdash (s_1 : A_1 \rightarrow B_1)(R \rightarrow S)(s_2 : A_2 \rightarrow B_2) \quad \Theta \vdash (t_1 : A_1)R(t_2 : A_2)}{\Theta \vdash (s_1 t_1 : B_1)S(s_2 t_2 : B_2)}$$

$$\frac{\Theta, \alpha\rho\beta \vdash (t_1 : A_1)R(t_2 : A_2)}{\Theta \vdash (\Lambda\alpha. t_1 : \forall\alpha. A_1)(\forall\alpha\rho\beta. R)(\Lambda\beta. t_2 : \forall\beta. A_2)}$$

$$\frac{\Theta \vdash (t_1 : \forall\alpha. A_1)(\forall\alpha\rho\beta. R)(t_2 : \forall\beta. A_2) \quad \Theta \vdash B_1SB_2 \text{ rel}}{\Theta \vdash (t_1[B_1] : A_1[\alpha \mapsto B_1])R[\alpha\rho\beta \mapsto B_1SB_2](t_2[B_2] : A_2[\beta \mapsto B_2])}$$

## Relatedness rules: standard relation formers

$$\frac{}{\Theta \vdash (x_1 : A_1)R(x_2 : A_2)} \quad ((x_1 : A_1)R(x_2 : A_2) \in \Theta)$$

$$\frac{\Theta, (x_1 : A_1)R(x_2 : A_2) \vdash (t_1 : B_1)S(t_2 : B_2)}{\Theta \vdash (\lambda x_1. t_1 : A_1 \rightarrow B_1)(R \rightarrow S)(\lambda x_2. t_2 : A_2 \rightarrow B_2)}$$

$$\frac{\Theta \vdash (s_1 : A_1 \rightarrow B_1)(R \rightarrow S)(s_2 : A_2 \rightarrow B_2) \quad \Theta \vdash (t_1 : A_1)R(t_2 : A_2)}{\Theta \vdash (s_1 t_1 : B_1)S(s_2 t_2 : B_2)}$$

$$\frac{\Theta, \alpha\beta \vdash (t_1 : A_1)R(t_2 : A_2)}{\Theta \vdash (\Lambda\alpha. t_1 : \forall\alpha. A_1)(\forall\alpha\beta. R)(\Lambda\beta. t_2 : \forall\beta. A_2)}$$

$$\frac{\Theta \vdash (t_1 : \forall\alpha. A_1)(\forall\alpha\beta. R)(t_2 : \forall\beta. A_2) \quad \Theta \vdash B_1SB_2 \text{ rel}}{\Theta \vdash (t_1[B_1] : A_1[\alpha \mapsto B_1])R[\alpha\beta \mapsto B_1SB_2](t_2[B_2] : A_2[\beta \mapsto B_2])}$$

## Relatedness rules: standard relation formers

$$\frac{}{\Theta \vdash (x_1 : A_1)R(x_2 : A_2)} \quad ((x_1 : A_1)R(x_2 : A_2) \in \Theta)$$

$$\frac{\Theta, (x_1 : A_1)R(x_2 : A_2) \vdash (t_1 : B_1)S(t_2 : B_2)}{\Theta \vdash (\lambda x_1. t_1 : A_1 \rightarrow B_1)(R \rightarrow S)(\lambda x_2. t_2 : A_2 \rightarrow B_2)}$$

$$\frac{\Theta \vdash (s_1 : A_1 \rightarrow B_1)(R \rightarrow S)(s_2 : A_2 \rightarrow B_2) \quad \Theta \vdash (t_1 : A_1)R(t_2 : A_2)}{\Theta \vdash (s_1 t_1 : B_1)S(s_2 t_2 : B_2)}$$

$$\frac{\Theta, \alpha\rho\beta \vdash (t_1 : A_1)R(t_2 : A_2)}{\Theta \vdash (\Lambda\alpha. t_1 : \forall\alpha. A_1)(\forall\alpha\rho\beta. R)(\Lambda\beta. t_2 : \forall\beta. A_2)}$$

$$\frac{\Theta \vdash (t_1 : \forall\alpha. A_1)(\forall\alpha\rho\beta. R)(t_2 : \forall\beta. A_2) \quad \Theta \vdash B_1SB_2 \text{ rel}}{\Theta \vdash (t_1[B_1] : A_1[\alpha \mapsto B_1])R[\alpha\rho\beta \mapsto B_1SB_2](t_2[B_2] : A_2[\beta \mapsto B_2])}$$

## Relatedness rules: inverse image relations and substitution

$$\frac{\Theta \vdash (t_1 u_1 : B_1)R(t_2 u_2 : B_2)}{\Theta \vdash (u_1 : A_1)([t_1 \times t_2]^{-1}R)(u_2 : A_2)}$$

$$\frac{\Theta \vdash (t_1 : A_1)R(t_2 : A_2) \quad \Theta_1 \vdash t_1 = s_1 : A_1 \quad \Theta_2 \vdash t_2 = s_2 : A_2}{\Theta \vdash (s_1 : A_1)R(s_2 : A_2)}$$

## One more rule: the parametricity rule

- The system get its power from inverse image relations together with the **parametricity rule**.
- Recall: If  $\Gamma \vdash t : A$  then  $\langle \Gamma \rangle \vdash (t : A) \langle A \rangle (t : A)$ .

## One more rule: the parametricity rule

- The system get its power from inverse image relations together with the **parametricity rule**.
- Recall: If  $\Gamma \vdash s = t : A$  then  $\langle \Gamma \rangle \vdash (s : A) \langle A \rangle (t : A)$ .

## One more rule: the parametricity rule

- The system get its power from inverse image relations together with the **parametricity rule**.
- Recall: If  $\Gamma \vdash s = t : A$  then  $\langle \Gamma \rangle \vdash (s : A) \langle A \rangle (t : A)$ .
- Parametricity rule states converse:

$$\frac{\langle \Gamma \rangle \vdash (s : A) \langle A \rangle (t : A)}{\Gamma \vdash s = t : A}$$

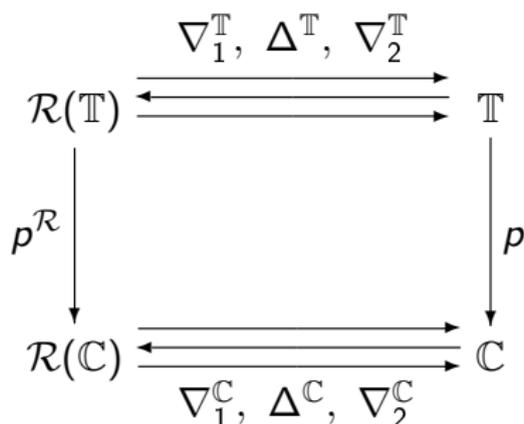
## One more rule: the parametricity rule

- The system get its power from inverse image relations together with the **parametricity rule**.
- Recall: If  $\Gamma \vdash s = t : A$  then  $\langle \Gamma \rangle \vdash (s : A) \langle A \rangle (t : A)$ .
- Parametricity rule states converse:

$$\frac{\langle \Gamma \rangle \vdash (s : A) \langle A \rangle (t : A)}{\Gamma \vdash s = t : A}$$

- So  $\langle A \rangle$  is the equality relation? No! Only in closed contexts.
- In fact, for open types,  $\langle A \rangle$  is not even a homogeneous relation, since  $\langle \alpha \rangle = \alpha \rho \beta$ .

# Interpretation in comprehensive $\lambda 2$ parametricity graphs



- $\lambda 2$  interpreted in  $p$ , as before.
- Relational context  $\Theta$  interpreted as an object  $[\Theta]$  in  $\mathcal{R}(\mathbb{C})$ .
- Syntactic relation  $\Theta \vdash ARB$  rel interpreted as a semantic relation  $[[R]]_{\Theta} : [[A]]_{(\Theta)_1} \leftrightarrow [[B]]_{(\Theta)_2}$  in  $\mathcal{R}(\mathbb{T})_{[[\Theta]]}$  using  $\lambda 2$  structure.

## Interpretation of inverse image relations

- Inverse-image relation  $\Theta \vdash A_1([t_1 \times t_2]^{-1}R)A_2$  rel interpreted using the *fibration* property of the parametricity graph:
- Have

$$\llbracket t_1 \rrbracket_{(\Theta)_1} : \mathbf{1} \longrightarrow \llbracket A_1 \rrbracket_{(\Theta)_1} \Rightarrow \llbracket B_1 \rrbracket_{(\Theta)_1}$$

$$\llbracket t_2 \rrbracket_{(\Theta)_2} : \mathbf{1} \longrightarrow \llbracket A_2 \rrbracket_{(\Theta)_2} \Rightarrow \llbracket B_2 \rrbracket_{(\Theta)_2}$$

## Interpretation of inverse image relations

- Inverse-image relation  $\Theta \vdash A_1([t_1 \times t_2]^{-1}R)A_2$  rel interpreted using the *fibration* property of the parametricity graph:
- Have

$$\llbracket t_1 \rrbracket_{(\Theta)_1} : \mathbf{1} \times \llbracket A_1 \rrbracket_{(\Theta)_1} \longrightarrow \llbracket B_1 \rrbracket_{(\Theta)_1}$$

$$\llbracket t_2 \rrbracket_{(\Theta)_2} : \mathbf{1} \times \llbracket A_2 \rrbracket_{(\Theta)_2} \longrightarrow \llbracket B_2 \rrbracket_{(\Theta)_2}$$

## Interpretation of inverse image relations

- Inverse-image relation  $\Theta \vdash A_1([t_1 \times t_2]^{-1}R)A_2$  rel interpreted using the *fibration* property of the parametricity graph:
- Have

$$\llbracket t_1 \rrbracket_{(\Theta)_1} : \llbracket A_1 \rrbracket_{(\Theta)_1} \longrightarrow \llbracket B_1 \rrbracket_{(\Theta)_1}$$

$$\llbracket t_2 \rrbracket_{(\Theta)_2} : \llbracket A_2 \rrbracket_{(\Theta)_2} \longrightarrow \llbracket B_2 \rrbracket_{(\Theta)_2}$$

# Interpretation of inverse image relations

- Inverse-image relation  $\Theta \vdash A_1([t_1 \times t_2]^{-1}R)A_2$  rel interpreted using the *fibration* property of the parametricity graph:
- Have

$$[[t_1]]_{(\Theta)_1} : [[A_1]]_{(\Theta)_1} \longrightarrow [[B_1]]_{(\Theta)_1}$$

$$[[t_2]]_{(\Theta)_2} : [[A_2]]_{(\Theta)_2} \longrightarrow [[B_2]]_{(\Theta)_2}$$

$$[[R]] : [[B_1]]_{(\Theta)_1} \leftrightarrow [[B_2]]_{(\Theta)_2}$$

# Interpretation of inverse image relations

- Inverse-image relation  $\Theta \vdash A_1([t_1 \times t_2]^{-1}R)A_2$  rel interpreted using the *fibration* property of the parametricity graph:

- Have

$$[[t_1]]_{(\Theta)_1} : [[A_1]]_{(\Theta)_1} \longrightarrow [[B_1]]_{(\Theta)_1}$$

$$[[t_2]]_{(\Theta)_2} : [[A_2]]_{(\Theta)_2} \longrightarrow [[B_2]]_{(\Theta)_2}$$

- Reindex  $[[R]] : [[B_1]]_{(\Theta)_1} \leftrightarrow [[B_2]]_{(\Theta)_2}$  in the fibration along these maps to interpret  $[[[t_1 \times t_2]^{-1}R]] : [[A_1]]_{(\Theta)_1} \leftrightarrow [[A_2]]_{(\Theta)_2}$ .

## Why didn't this work before?

- If we try to replay the interpretation in the old-fashioned semantics without comprehension, we get:

$$\llbracket t_1 \rrbracket' : (\llbracket \Delta \rrbracket)_1 \longrightarrow (\llbracket A_1 \rrbracket)_1 \Rightarrow (\llbracket B_1 \rrbracket)_1$$

$$\llbracket t_2 \rrbracket' : (\llbracket \Delta \rrbracket)_2 \longrightarrow (\llbracket A_2 \rrbracket)_2 \Rightarrow (\llbracket B_2 \rrbracket)_2$$

## Why didn't this work before?

- If we try to replay the interpretation in the old-fashioned semantics without comprehension, we get:

$$\llbracket t_1 \rrbracket' : (\llbracket \Delta \rrbracket)_1 \times (\llbracket A_1 \rrbracket)_1 \longrightarrow (\llbracket B_1 \rrbracket)_1$$

$$\llbracket t_2 \rrbracket' : (\llbracket \Delta \rrbracket)_2 \times (\llbracket A_2 \rrbracket)_2 \longrightarrow (\llbracket B_2 \rrbracket)_2$$

## Why didn't this work before?

- If we try to replay the interpretation in the old-fashioned semantics without comprehension, we get:

$$\llbracket t_1 \rrbracket' : (\llbracket \Delta \rrbracket)_1 \times (\llbracket A_1 \rrbracket)_1 \longrightarrow (\llbracket B_1 \rrbracket)_1$$

$$\llbracket t_2 \rrbracket' : (\llbracket \Delta \rrbracket)_2 \times (\llbracket A_2 \rrbracket)_2 \longrightarrow (\llbracket B_2 \rrbracket)_2$$

- Reindexing along this does not give a relation  $(\llbracket A_1 \rrbracket)_1 \leftrightarrow (\llbracket A_2 \rrbracket)_2!$

## Why didn't this work before?

- If we try to replay the interpretation in the old-fashioned semantics without comprehension, we get:

$$\llbracket t_1 \rrbracket' : (\llbracket \Delta \rrbracket)_1 \times (\llbracket A_1 \rrbracket)_1 \longrightarrow (\llbracket B_1 \rrbracket)_1$$

$$\llbracket t_2 \rrbracket' : (\llbracket \Delta \rrbracket)_2 \times (\llbracket A_2 \rrbracket)_2 \longrightarrow (\llbracket B_2 \rrbracket)_2$$

- Reindexing along this does not give a relation  $(\llbracket A_1 \rrbracket)_1 \leftrightarrow (\llbracket A_2 \rrbracket)_2!$
- So things work because in the new semantics,  $\llbracket t_i \rrbracket_{(\Theta)_i}$  are **global points**. Possible because of use of **comprehension**.

# Soundness

## Theorem (Soundness for $\lambda 2R$ )

*In every comprehensive  $\lambda 2$  parametricity graph:*

- 1 *if  $\Gamma \vdash t_1 = t_2 : A$  then  $\llbracket t_1 \rrbracket_\Gamma = \llbracket t_2 \rrbracket_\Gamma$ ; and*
- 2 *if  $\Theta \vdash (t_1 : A_1)R(t_2 : A_2)$  then  $\llbracket t_1 \rrbracket_{(\Theta)_1} \times \llbracket t_2 \rrbracket_{(\Theta)_2} : \mathbf{1}_{\llbracket \Theta \rrbracket} \longrightarrow \llbracket R \rrbracket_\Theta$ .*

# Soundness

## Theorem (Soundness for $\lambda 2R$ )

In every comprehensive  $\lambda 2$  parametricity graph:

- 1 if  $\Gamma \vdash t_1 = t_2 : A$  then  $\llbracket t_1 \rrbracket_\Gamma = \llbracket t_2 \rrbracket_\Gamma$ ; and
- 2 if  $\Theta \vdash (t_1 : A_1)R(t_2 : A_2)$  then  $\llbracket t_1 \rrbracket_{(\Theta)_1} \times \llbracket t_2 \rrbracket_{(\Theta)_2} : \mathbf{1}_{\llbracket \Theta \rrbracket} \longrightarrow \llbracket R \rrbracket_\Theta$ .

Substitution in relations sound by *relational* property.

Parametricity rule sound by *identity* property.

Inverse image rules sound by *fibration* property.

## ... and completeness

### Theorem (Full completeness for $\lambda 2R$ )

There exists a comprehensive  $\lambda 2$  parametricity graph satisfying the following.

- 1 For every type  $\Gamma \vdash A$  type, every global point  $\mathbf{1}_{\llbracket \Gamma \rrbracket} \longrightarrow \llbracket A \rrbracket_{\Gamma}$  is the denotation  $\llbracket t \rrbracket_{\Gamma}$  of some term  $\Gamma \vdash t : A$ .
- 2 For all terms  $\Gamma \vdash t_1, t_2 : A$  satisfying  $\llbracket t_1 \rrbracket_{\Gamma} = \llbracket t_2 \rrbracket_{\Gamma}$ , we have  $\Gamma \vdash t_1 = t_2 : A$ .
- 3 For every relation  $\Theta \vdash A_1 R A_2$  type, every global point  $\mathbf{1}_{\llbracket \Theta \rrbracket} \longrightarrow \llbracket R \rrbracket_{\Theta}$  arises as  $\llbracket t_1 \rrbracket_{(\Theta)_1} \times \llbracket t_2 \rrbracket_{(\Theta)_2}$  for terms  $t_1, t_2$  such that  $\Theta \vdash (t_1 : A_1) R (t_2 : A_2)$ .

A stack of several old, worn books with leather and cloth covers. The spines are visible, showing signs of age and use. Some spines have gold lettering, including the word "LEDGER". Small white labels with handwritten numbers are attached to the spines. A semi-transparent text box is overlaid on the image.

Deriving the expected consequences

01-2-10

01-2-10

01-2-10

Warm-up:  $\forall\alpha. \alpha \rightarrow \alpha$  is terminal

- Want to prove  $\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha \vdash z = \Lambda\alpha. \lambda x. x : \forall\alpha. \alpha \rightarrow \alpha$ .

## Warm-up: $\forall\alpha. \alpha \rightarrow \alpha$ is terminal

- Want to prove  $\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha \vdash z = \Lambda\alpha. \lambda x. x : \forall\alpha. \alpha \rightarrow \alpha$ .
- By extensionality, it is enough to show

$$\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha, \alpha, x : \alpha \vdash z[\alpha] x = x : \alpha$$

## Warm-up: $\forall\alpha. \alpha \rightarrow \alpha$ is terminal

- Want to prove  $\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha \vdash z = \Lambda\alpha. \lambda x. x : \forall\alpha. \alpha \rightarrow \alpha$ .
- By extensionality, it is enough to show

$$\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha, \alpha, x : \alpha \vdash z[\alpha] x = x : \alpha$$

- Further by the parametricity rule, it is enough to show

$$\langle \Gamma, z : \forall\alpha. \alpha \rightarrow \alpha, \alpha, x : \alpha \rangle \vdash (z[\alpha] x : \alpha) \langle \alpha \rangle (x : \alpha)$$

## Warm-up: $\forall\alpha. \alpha \rightarrow \alpha$ is terminal

- Want to prove  $\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha \vdash z = \Lambda\alpha. \lambda x. x : \forall\alpha. \alpha \rightarrow \alpha$ .
- By extensionality, it is enough to show

$$\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha, \alpha, x : \alpha \vdash z[\alpha] x = x : \alpha$$

- Further by the parametricity rule, it is enough to show

$$\langle \Gamma \rangle, z(\forall\alpha\rho\beta. \rho \rightarrow \rho) w, \alpha\rho\beta, (x : \alpha)\rho(y : \beta) \vdash (z[\alpha] x : \alpha)\rho(y : \beta)$$

## Warm-up: $\forall\alpha. \alpha \rightarrow \alpha$ is terminal

- Want to prove  $\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha \vdash z = \Lambda\alpha. \lambda x. x : \forall\alpha. \alpha \rightarrow \alpha$ .
- By extensionality, it is enough to show

$$\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha, \alpha, x : \alpha \vdash z[\alpha] x = x : \alpha$$

- Further by the parametricity rule, it is enough to show

$$\langle \Gamma \rangle, z(\forall\alpha\rho\beta. \rho \rightarrow \rho) w, \alpha\rho\beta, (x : \alpha)\rho(y : \beta) \vdash (z[\alpha] x : \alpha)\rho(y : \beta)$$

- $(x : \alpha)R(w : \forall\alpha. \alpha \rightarrow \alpha)$  where  $R = ([\text{id} \times (\lambda_. y)]^{-1}\rho)$ , since  $x\rho y$ .

## Warm-up: $\forall\alpha. \alpha \rightarrow \alpha$ is terminal

- Want to prove  $\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha \vdash z = \Lambda\alpha. \lambda x. x : \forall\alpha. \alpha \rightarrow \alpha$ .
- By extensionality, it is enough to show

$$\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha, \alpha, x : \alpha \vdash z[\alpha] x = x : \alpha$$

- Further by the parametricity rule, it is enough to show

$$\langle \Gamma \rangle, z(\forall\alpha\rho\beta. \rho \rightarrow \rho) w, \alpha\rho\beta, (x : \alpha)\rho(y : \beta) \vdash (z[\alpha] x : \alpha)\rho(y : \beta)$$

- $(x : \alpha)R(w : \forall\alpha. \alpha \rightarrow \alpha)$  where  $R = ([\text{id} \times (\lambda_. y)]^{-1}\rho)$ , since  $x\rho y$ .
- Since  $z(\forall\rho. \rho \rightarrow \rho) w$ , by instantiating  $\alpha\rho\beta = \alpha R(\forall\beta. \beta \rightarrow \beta)$

$$(z[\alpha])(R \rightarrow R)(w[\forall\beta. \beta \rightarrow \beta])$$

## Warm-up: $\forall\alpha. \alpha \rightarrow \alpha$ is terminal

- Want to prove  $\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha \vdash z = \Lambda\alpha. \lambda x. x : \forall\alpha. \alpha \rightarrow \alpha$ .
- By extensionality, it is enough to show

$$\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha, \alpha, x : \alpha \vdash z[\alpha] x = x : \alpha$$

- Further by the parametricity rule, it is enough to show

$$\langle \Gamma \rangle, z(\forall\alpha\rho\beta. \rho \rightarrow \rho) w, \alpha\rho\beta, (x : \alpha)\rho(y : \beta) \vdash (z[\alpha] x : \alpha)\rho(y : \beta)$$

- $(x : \alpha)R(w : \forall\alpha. \alpha \rightarrow \alpha)$  where  $R = ([\text{id} \times (\lambda_. y)]^{-1}\rho)$ , since  $x\rho y$ .
- Since  $z(\forall\rho. \rho \rightarrow \rho)w$ , by instantiating  $\alpha\rho\beta = \alpha R(\forall\beta. \beta \rightarrow \beta)$

$$(z[\alpha])(R \rightarrow R)(w[\forall\beta. \beta \rightarrow \beta])$$

hence

$$(z[\alpha] x)R(w[\forall\beta. \beta \rightarrow \beta] w)$$

## Warm-up: $\forall\alpha. \alpha \rightarrow \alpha$ is terminal

- Want to prove  $\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha \vdash z = \Lambda\alpha. \lambda x. x : \forall\alpha. \alpha \rightarrow \alpha$ .
- By extensionality, it is enough to show

$$\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha, \alpha, x : \alpha \vdash z[\alpha] x = x : \alpha$$

- Further by the parametricity rule, it is enough to show

$$\langle \Gamma \rangle, z(\forall\alpha\rho\beta. \rho \rightarrow \rho) w, \alpha\rho\beta, (x : \alpha)\rho(y : \beta) \vdash (z[\alpha] x : \alpha)\rho(y : \beta)$$

- $(x : \alpha)R(w : \forall\alpha. \alpha \rightarrow \alpha)$  where  $R = ([\text{id} \times (\lambda\_ . y)]^{-1}\rho)$ , since  $x\rho y$ .
- Since  $z(\forall\rho. \rho \rightarrow \rho) w$ , by instantiating  $\alpha\rho\beta = \alpha R(\forall\beta. \beta \rightarrow \beta)$

$$(z[\alpha])(R \rightarrow R)(w[\forall\beta. \beta \rightarrow \beta])$$

hence

$$(z[\alpha] x)([\text{id} \times (\lambda\_ . y)]^{-1}\rho)(w[\forall\beta. \beta \rightarrow \beta] w)$$

## Warm-up: $\forall\alpha. \alpha \rightarrow \alpha$ is terminal

- Want to prove  $\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha \vdash z = \Lambda\alpha. \lambda x. x : \forall\alpha. \alpha \rightarrow \alpha$ .
- By extensionality, it is enough to show

$$\Gamma, z : \forall\alpha. \alpha \rightarrow \alpha, \alpha, x : \alpha \vdash z[\alpha] x = x : \alpha$$

- Further by the parametricity rule, it is enough to show

$$\langle \Gamma \rangle, z(\forall\alpha\rho\beta. \rho \rightarrow \rho) w, \alpha\rho\beta, (x : \alpha)\rho(y : \beta) \vdash (z[\alpha] x : \alpha)\rho(y : \beta)$$

- $(x : \alpha)R(w : \forall\alpha. \alpha \rightarrow \alpha)$  where  $R = ([\text{id} \times (\lambda\_ . y)]^{-1}\rho)$ , since  $x\rho y$ .
- Since  $z(\forall\rho. \rho \rightarrow \rho)w$ , by instantiating  $\alpha\rho\beta = \alpha R(\forall\beta. \beta \rightarrow \beta)$

$$(z[\alpha])(R \rightarrow R)(w[\forall\beta. \beta \rightarrow \beta])$$

hence

$$(z[\alpha] x)([\text{id} \times (\lambda\_ . y)]^{-1}\rho)(w[\forall\beta. \beta \rightarrow \beta] w)$$

i.e.

$$(z[\alpha] x : \alpha)\rho(y : \beta).$$

# The expected consequences

## Theorem (Consequences of Parametricity)

System  $\lambda 2R$  proves:

- 1  $\forall \alpha. \alpha \rightarrow \alpha$  is **1**.
- 2  $\forall \alpha. (A \rightarrow B \rightarrow \alpha) \rightarrow \alpha$  is  $A \times B$ .
- 3  $\forall \alpha. \alpha$  is **0**.
- 4  $\forall \alpha. (A \rightarrow \alpha) \rightarrow (B \rightarrow \alpha) \rightarrow \alpha$  is  $A + B$ .
- 5  $\forall \alpha. (\forall \beta. (T(\beta) \rightarrow \alpha)) \rightarrow \alpha$  is  $\exists \alpha. T(\alpha)$ .
- 6 The type  $\forall \alpha. (T(\alpha) \rightarrow \alpha) \rightarrow \alpha$  is the carrier of the initial  $T$ -algebra for all functorial type expressions  $T(\alpha)$ .
- 7 The type  $\exists \alpha. (\alpha \rightarrow T(\alpha)) \times \alpha$  is the carrier of the final  $T$ -coalgebra for all functorial type expressions  $T(\alpha)$ .
- 8 Terms of type  $\forall \alpha. F(\alpha, \alpha) \rightarrow G(\alpha, \alpha)$  for mixed-variance type expressions  $F$  and  $G$  are dinatural.

## Some comments on the proof

- As usual, relations representing **graphs of functions** play a key role.

## Some comments on the proof

- As usual, relations representing **graphs of functions** play a key role.
- Two ways to define concrete graphs:
  - ▶  $(x:A) gr_*(f) (y:B)$  if  $f x = y$ .

## Some comments on the proof

- As usual, relations representing **graphs of functions** play a key role.
- Two ways to define concrete graphs:
  - ▶  $(x:A) gr_*(f)(y:B)$  if  $f x = y$ .
  - ▶  $(x:A) gr_1(f)(y:B)$  if there exists  $w:A$  such that  $x = w$  and  $y = f w$ .

## Some comments on the proof

- As usual, relations representing **graphs of functions** play a key role.
- Two ways to define concrete graphs:
  - ▶  $(x:A)gr_*(f)(y:B)$  if  $(f x:B)\langle B\rangle(y:B)$ .
  - ▶  $(x:A)gr_1(f)(f w:B)$  if there exists  $w:A$  such that  $(x:A)\langle A\rangle(w:A)$ .
  - ▶ Since we only have pseudo-identities, these **do not coincide** in general.

## Some comments on the proof

- As usual, relations representing **graphs of functions** play a key role.
- Two ways to define concrete graphs:
  - ▶  $(x:A)gr_*(f)(y:B)$  if  $(f x:B)\langle B\rangle(y:B)$ .
  - ▶  $(x:A)gr_!(f)(f w:B)$  if there exists  $w:A$  such that  $(x:A)\langle A\rangle(w:A)$ .
  - ▶ Since we only have pseudo-identities, these **do not coincide** in general.
- $gr_*(f) := [f \times \text{id}]^{-1}\langle B\rangle$  defined using fibrational structure,  
 $gr_!(f) := [\text{id} \times f]_!\langle A\rangle$  using derived opfibrational structure.

## Some comments on the proof

- As usual, relations representing **graphs of functions** play a key role.
- Two ways to define concrete graphs:
  - ▶  $(x:A)gr_*(f)(y:B)$  if  $(f x:B)\langle B\rangle(y:B)$ .
  - ▶  $(x:A)gr_!(f)(f w:B)$  if there exists  $w:A$  such that  $(x:A)\langle A\rangle(w:A)$ .
  - ▶ Since we only have pseudo-identities, these **do not coincide** in general.
- $gr_*(f) := [f \times \text{id}]^{-1}\langle B\rangle$  defined using fibrational structure,  
 $gr_!(f) := [\text{id} \times f]_!\langle A\rangle$  using derived opfibrational structure.
- Subtlety: initial algebras use inverse image pseudographs, final coalgebras direct image ones.

A photograph of a rustic library interior. The room features high wooden ceilings, a staircase with wooden railings on the left, and floor-to-ceiling wooden bookshelves filled with books on the right. A large, glowing spherical lamp hangs from the ceiling, and a stained glass window is visible in the background. The overall atmosphere is warm and inviting.

# Summary

## Summary

- $\lambda 2$  fibrations with **comprehension property** as natural models of  $\lambda 2$  (sound and complete).
- Comprehensive  $\lambda 2$  **parametricity graphs** form good models of relational parametricity for  $\lambda 2$ , with usual strong consequences.
- Reasoning in the models using a sound and complete type theory  $\lambda 2R$ , including **inverse image relations**.
- Proof of consequences of parametricity involves novel ingredients:
  - ▶ **direct image relations** via impredicative encoding,
  - ▶ **no identity relations** available, and
  - ▶ two different **pseudo-graph** relations (using inverse and direct images).
- **Future work:** Extend to e.g. dependent type theory.



Neil Ghani, Fredrik Nordvall Forsberg and Alex Simpson

Comprehensive parametric polymorphism: categorical models and type theory.

FoSSaCS 2016.

# Summary

- $\lambda 2$  filter (sound)
- Comprehensiveness of relative
- Reasoning including
- Proof
- ▶
- ▶
- ▶
- **Futu**



Neil

Comprehensive parametric polymorphism, categorical models and type theory.

FoSSaCS 2016.