Comprehensive parametric polymorphism

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Joint work with...

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Parametric polymorphism [Strachey, 1967]

- A polymorphic program
$$t : \forall\alpha.\, A$$
is parametric if it applies the same uniform algorithm at all instantiations $t[B]$ of its type parameter.

- Typical example:
$$\texttt{reverse} : \forall\alpha.\, \texttt{List}\ \alpha \to \texttt{List}\ \alpha$$

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- Turn the negative statement "not distinguishing types" into the positive statement "preserves all relations".

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- Turn the negative statement "not distinguishing types" into the positive statement "preserves all relations".
- A polymorphic program $t : \forall\alpha. A$ is relationally parametric if for all relations $R \subseteq B \times B'$,

  $$(t[B], t[B']) \in \langle A\rangle(R)$$

  where $\langle A\rangle(R) \subseteq A(B) \times A(B')$ is the relational interpretation of the type A.
- E.g. `reverse` $: \forall\alpha.\mathtt{List}\ \alpha \to \mathtt{List}\ \alpha$ is relationally parametric.

Applications of relational parametricity

Relational parametricity enables:

- Reasoning about abstract data types.
- Correctness (universal properties) of encodings of data types.
- 'Theorems for free!' [Wadler, 1989].
- Concretely, a specific example: if $t : \forall\alpha.\, \alpha \to \alpha$ then $t = \Lambda\alpha.\, \lambda x.\, x$.

Usually in the setting of Girard's/Reynold's $\lambda\mathbf{2}$ (System F) — serves as a model type theory for (impredicative) polymorphism.

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- We know the fundamental structure needed for $\lambda\mathbf{2}$ ($\lambda\mathbf{2}$ fibrations [Seely, 1987]).
- We also know the fundamental structures used for relational parametricity (reflexive graph categories [Robinson and Rosolini, 1994], parametricity graphs [Dunphy and Reddy, 2004]).

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- When doing so, the expected consequences of parametricity are only derivable if the underlying category is well-pointed.
- Recall: A category $\mathbb{C}$ is well-pointed when $f = g : A \longrightarrow B$ in $\mathbb{C}$ iff $f \circ e = g \circ e : \mathbf{1} \longrightarrow B$ for all global elements $e : \mathbf{1} \longrightarrow A$.
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- This rules out many interesting categories, e.g. functor categories.

- Existing solutions (e.g. Birkedal and Møgelberg [2005]) circumvent this by adding significant additional structure to models (enough to model the full logic of Plotkin and Abadi).

- We seek instead a mininimal solution still based on the idea of directly combining models of $\lambda\mathbf{2}$ with structure for relational parametricity.

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- We achieve this in a perhaps unexpected way: we change the notion of model of $\lambda\mathbf{2}$.
- $\lambda\mathbf{2}$ fibrations satisfying Lawvere's comprehension property.
- This allows us to combine such comprehensive $\lambda\mathbf{2}$ fibrations with reflexive graph structure to model relational parametricity for $\lambda\mathbf{2}$.
- Validating expected consequences, also for non-well-pointed categories.
- Proof involves novel ingredients due to minimality of structure:
  - definability of direct image relations,
  - arguments without use of equality relations, and
  - only weak forms of graph relations available ('pseudographs').

Outline

1. The type theory $\lambda\mathbf{2}$
2. Modelling $\lambda\mathbf{2}$ using (comprehensive) $\lambda\mathbf{2}$ fibrations
3. Modelling relational parametricity using (comprehensive) parametricity graphs
4. Reasoning about parametricity using a type theory $\lambda\mathbf{2R}$

The type theory $\lambda 2$

The polymorphic lambda calculus $\lambda\mathbf{2}$ (System F) [Girard, 1972; Reynolds, 1974]

- Four judgements:

$$
\begin{array}{rl}
\Gamma \text{ ctxt} & \Gamma \text{ is a context} \\
\Gamma \vdash A \text{ type} & A \text{ is a type in context } \Gamma \\
\Gamma \vdash t : A & \text{term } t \text{ has type } A \text{ in context } \Gamma \\
\Gamma \vdash t = s : A & \text{judgemental equality}
\end{array}
$$

- Types and terms generated by grammars

$$
\begin{array}{rlr}
A, B & ::= \alpha \mid A \to B \mid \forall \alpha.\, A & \text{types} \\
t, s & ::= x \mid \lambda x.\, t \mid t \; s \mid \Lambda \alpha.\, t \mid t[B] & \text{terms}
\end{array}
$$

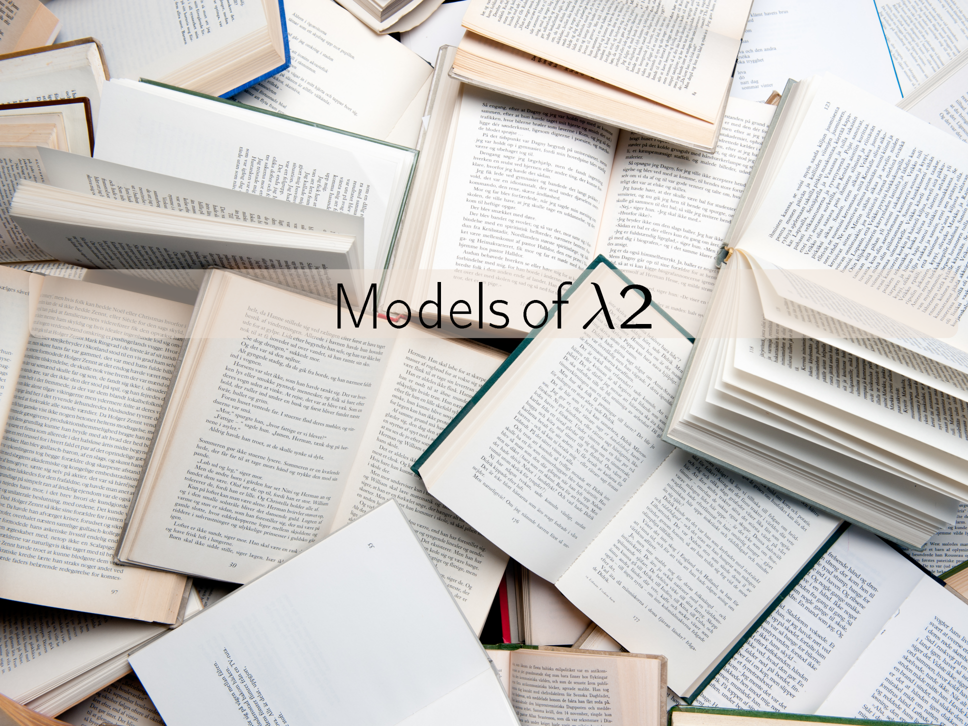
- Equality generated by (β) and (η) for both term and type abstraction.

Only unusual feature of our presentation

- We use a single context with type and term variables interleaved.
- Standard from a dependent types perspective.
- Hence two different context extensions:

$$\frac{\Gamma \ \mathsf{ctxt}}{\Gamma, \alpha \ \mathsf{ctxt}} \ (\alpha \notin \Gamma) \qquad\qquad \frac{\Gamma \ \mathsf{ctxt} \qquad \Gamma \vdash A \ \mathsf{type}}{\Gamma, x : A \ \mathsf{ctxt}} \ (x \notin \Gamma)$$

Models of $\lambda 2$



$\lambda 2$ fibrations [Seely, 1987; see also Jacobs, 1999]

Definition ($\lambda 2$ fibration)

A $\lambda\mathbf{2}$ fibration is a fibration $p : \mathbb{T} \to \mathbb{C}$, where the base category $\mathbb{C}$ has finite products, and the fibration:

1. is fibred cartesian closed;
2. has a generic object U — we write Ω for $p\,U$;
3. and has fibred-products along projections $X \times \Omega \longrightarrow X$ in $\mathbb{C}$.

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A $\lambda 2$ **fibration** is a split fibration $p : \mathbb{T} \to \mathbb{C}$, where the base category $\mathbb{C}$ has finite products, and the fibration:

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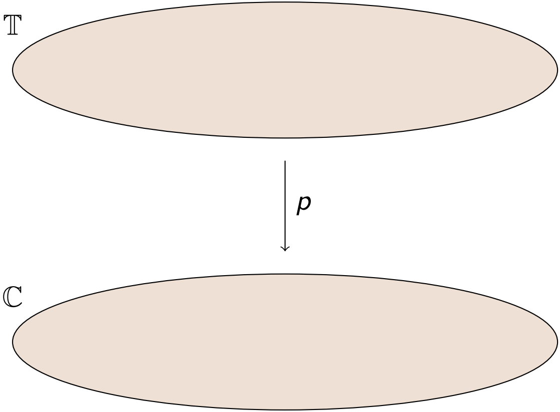
2. has a split generic object U — we write Ω for $p\,U$;
3. and has fibred-products along projections $X \times \Omega \longrightarrow X$ in $\mathbb{C}$.

Moreover, the reindexing functors given by the splitting should preserve the above-specified structure in fibres on the nose.

Structure in detail (i)

- Fibration $p : \mathbb{T} \to \mathbb{C}$, $\mathbb{C}$ has finite products.
  - $\mathbb{C}$ category of type variable contexts and substitutions.
  - Products are context concatenation.



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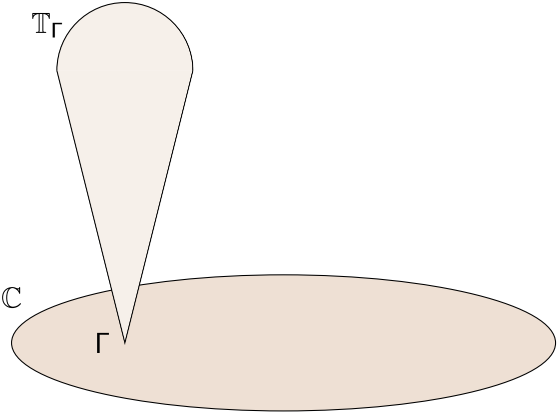

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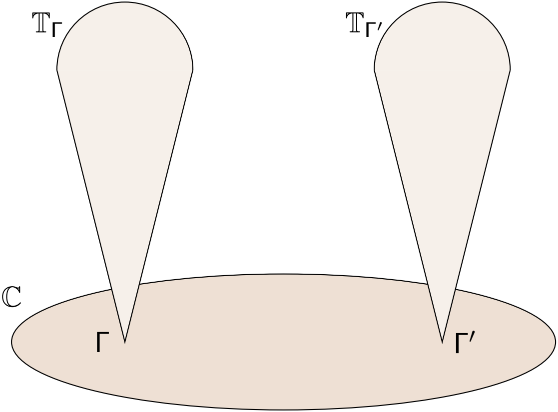

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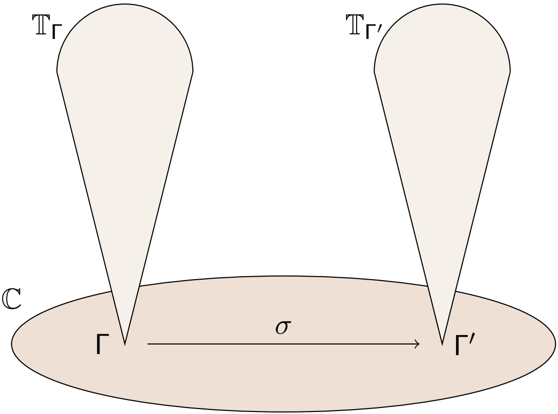

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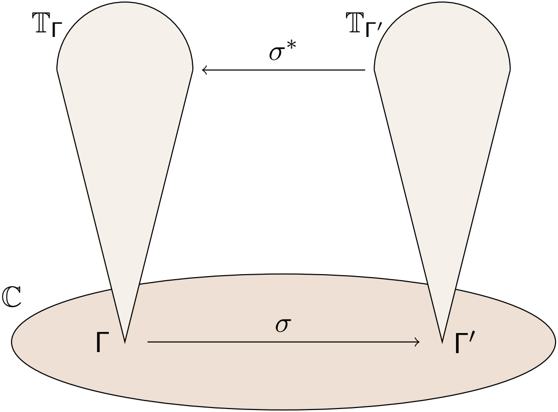
  - Fibre $\mathbb{T}_\Gamma$ category of types in context Γ.
  - Reindexing is substitution.



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- . . . and has fibred-products along projections $\Gamma \times \Omega \longrightarrow \Gamma$ in $\mathbb{C}$.
  - Each reindexing functor $\pi_\Omega^* : \mathbb{T}_\Gamma \to \mathbb{T}_{\Gamma\times\Omega}$ has a right adjoint $\prod_\Omega : \mathbb{T}_{\Gamma\times\Omega} \to \mathbb{T}_\Gamma$.
  - Needed for $\forall$.

Old-fashioned interpretation

- Given context Γ, let $\Theta = \alpha_1, \ldots, \alpha_n$ and $\Delta = x_1 : A_1, \ldots, x_m : A_m$ be the type and term variable components of Γ.

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- The combined context made things awkward; let's fix that by modifying the notion of model and giving a new interpretation.

Our modification: one new ingredient

We take inspirations from models of dependent types, where separated contexts are not possible.

Definition (Comprehensive $\lambda 2$ fibration)

A $\lambda\mathbf{2}$ fibration $p : \mathbb{T} \to \mathbb{C}$ is comprehensive if it enjoys the comprehension property: the fibred-terminal-object functor $X \mapsto \mathbf{1}_X : \mathbb{C} \to \mathbb{T}$ has a specified right adjoint $K : \mathbb{T} \to \mathbb{C}$.

- Given $A \in \mathbb{T}_\Gamma$, think of $K(A)$ as the extended context $\Gamma, x : A$.
- For $A \in \mathbb{T}_\Gamma$, write $\kappa_A = p(\varepsilon_A) : K(A) \longrightarrow \Gamma$ for the 'projection' map obtained by applying p to the counit $\varepsilon_A : \mathbf{1}_{K(A)} \longrightarrow A$ in $\mathbb{T}$.

Interpretation in a comprehensive $\lambda\mathbf{2}$ fibration

- Contexts Γ interpreted as object $[\![\Gamma]\!]$ in $\mathbb{C}$.

- Type $\Gamma \vdash A$ type interpreted as object $[\![A]\!]_\Gamma$ in $\mathbb{T}_{[\![\Gamma]\!]}$.

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- Mutually defined, simultaneously with maps $\pi^\alpha_\Gamma : [\![\Gamma]\!] \longrightarrow \Omega$ for every context Γ containing α.

$$
\begin{aligned}
[\![\cdot]\!] &= \mathbf{1} & [\![\alpha]\!]_\Gamma &= (\pi^\alpha_\Gamma)^*\, U \\
[\![\Gamma, \alpha]\!] &= [\![\Gamma]\!] \times \Omega & [\![A \to B]\!]_\Gamma &= [\![A]\!]_\Gamma \Rightarrow_{[\![\Gamma]\!]} [\![B]\!]_\Gamma \\
[\![\Gamma, x : A]\!] &= K[\![A]\!]_\Gamma & [\![\forall \alpha.\, A]\!] &= \textstyle\prod_\Omega [\![A]\!]_{\Gamma, \alpha}
\end{aligned}
$$

$$
\pi^\alpha_{\Gamma, \alpha} = \pi_2 \qquad \pi^\alpha_{\Gamma, \beta} = \pi^\alpha_\Gamma \circ \pi_1 \ (\beta \neq \alpha) \qquad \pi^\alpha_{\Gamma, x:A} = \pi^\alpha_\Gamma \circ \kappa_{[\![A]\!]_\Gamma}
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$$[\![\Gamma, x : A]\!] = K[\![A]\!]_\Gamma \qquad\qquad [\![\forall\alpha.\, A]\!] = \prod\nolimits_\Omega [\![A]\!]_{\Gamma,\alpha}$$

$$\pi^\alpha_{\Gamma,\alpha} = \pi_2 \qquad \pi^\alpha_{\Gamma,\beta} = \pi^\alpha_\Gamma \circ \pi_1 \ (\beta \neq \alpha) \qquad \pi^\alpha_{\Gamma, x:A} = \pi^\alpha_\Gamma \circ \kappa_{[\![A]\!]_\Gamma}$$

- Term $\Gamma \vdash t : A$ is interpreted as global element

$$[\![t]\!]_\Gamma : \mathbf{1}_{[\![\Gamma]\!]} \longrightarrow [\![A]\!]_\Gamma \qquad \text{in } \mathbb{T}_{[\![\Gamma]\!]}$$

For future reference

Compare the interpretation of terms in standard and comprehensive $\lambda\mathbf{2}$ fibrations:

- $[\![t]\!]_{\Theta;\Delta} : [\![\Delta]\!] \to [\![A]\!]$ in $\mathbb{T}_{[\![\Theta]\!]}$ (old-fashioned, standard)
- versus global element

$$[\![t]\!]_\Gamma : \mathbf{1}_{[\![\Gamma]\!]} \longrightarrow [\![A]\!]_\Gamma \qquad \text{in } \mathbb{T}_{[\![\Gamma]\!]}$$

(comprehensive)

Soundness and completeness

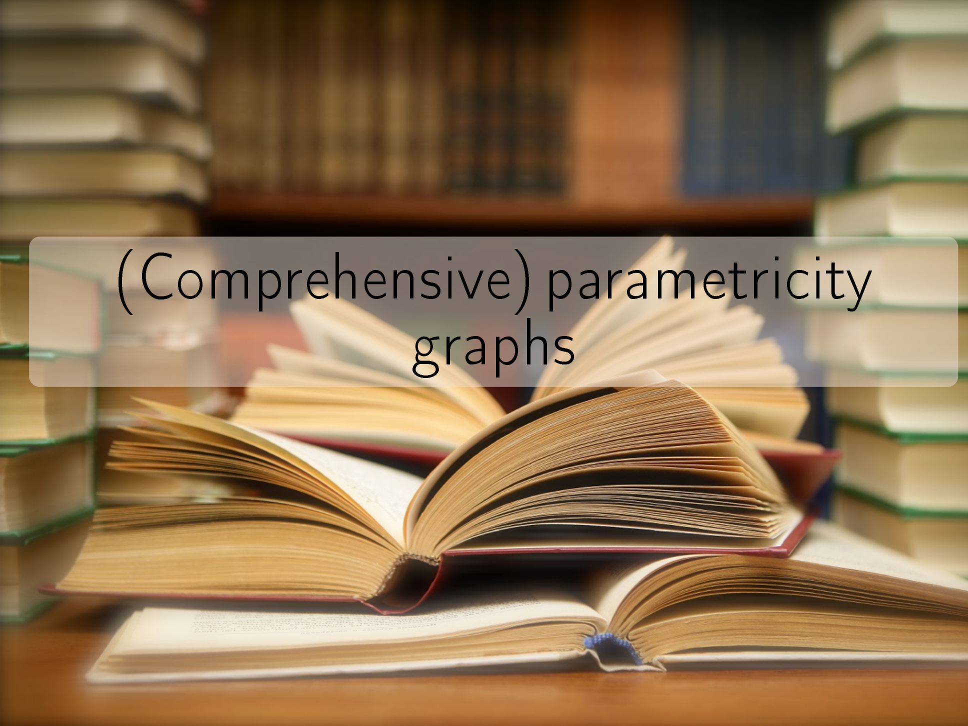
Theorem (Soundness for $\lambda\mathbf{2}$)

If $\Gamma \vdash t_1 = t_2 : A$ then, in every comprehensive $\lambda\mathbf{2}$ fibration, we have $[\![t_1]\!]_\Gamma = [\![t_2]\!]_\Gamma$.

Theorem (Full completeness for $\lambda\mathbf{2}$)

There exists a comprehensive $\lambda\mathbf{2}$ fibration satisfying:

1. *for every type $\Gamma \vdash A$ type, every global point $\mathbf{1}_{[\![\Gamma]\!]} \longrightarrow [\![A]\!]_\Gamma$ is the denotation $[\![t]\!]_\Gamma$ of some term $\Gamma \vdash t : A$; and*
2. *for all terms $\Gamma \vdash t_1, t_2 : A$ satisfying $[\![t_1]\!]_\Gamma = [\![t_2]\!]_\Gamma$, we have $\Gamma \vdash t_1 = t_2 : A$.*



(Comprehensive) parametricity graphs

Incorporating relational parametricity

- These models do not model parametricity.
- In order to do so, we combine with the structure of reflexive graph categories [Ma and Reynolds,1992; Robinson and Rosolini, 1994; O'Hearn and Tennent, 1995; . . .].
- Simple category-theoretic structure for modelling relations.

Reflexive graph categories

$$\begin{tikzcd} \mathbb{E} \arrow[r, shift left=2, "\nabla_1"] \arrow[r, shift right=2, "\nabla_2"'] & \mathbb{V} \arrow[l, "\Delta" description] \end{tikzcd}$$

- Categories $\mathbb{V}$ and $\mathbb{E}$, where we think if $\mathbb{E}$ as category of relations over objects of $\mathbb{V}$.
- The functors ∇_1, ∇_2 are 'projection' functors giving source and target of relations, respectively, and Δ maps an object to its 'identity relation'.

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- Notation: $R\colon A \leftrightarrow B$ means $R \in \mathbb{E}$ and $\nabla_1 R = A$, $\nabla_2 R = B$.

- Similarly, write $f \times g\colon R \longrightarrow S$ if there is $h\colon R \longrightarrow S$ in $\mathbb{E}$ with $\nabla_1 h = f$ and $\nabla_2 h = g$. (Will soon assume h is unique, if it exists.)

Parametricity graphs [Dunphy, 2002; Dunphy and Reddy, 2004]

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- Identity property if for every $h : \Delta A \longrightarrow \Delta B$ in $\mathbb{E}$, it holds that $\nabla_1 h = \nabla_2 h$. Allows one to think of ΔA as an identity relation on A.
- Parametricity graph: relational, with the identity property, and $\langle \nabla_1, \nabla_2 \rangle : \mathbb{E} \to \mathbb{V} \times \mathbb{V}$ a fibration. Ensures that there are enough relations by supplying inverse image relations.

Combining reflexive graphs and comprehensive $\lambda\mathbf{2}$ fibrations

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Main definition (Comprehensive $\lambda 2$ parametricity graph)

A comprehensive $\lambda 2$ parametricity graph is a reflexive graph of comprehensive $\lambda 2$ fibrations

$$\begin{tikzcd}
\mathcal{R}(\mathbb{T}) \arrow[r, shift left=2, "{\nabla_1^{\mathbb{T}},\ \Delta^{\mathbb{T}},\ \nabla_2^{\mathbb{T}}}"] \arrow[r, shift right=2] \arrow[d, "p^{\mathcal{R}}"'] & \mathbb{T} \arrow[l] \arrow[d, "p"] \\
\mathcal{R}(\mathbb{C}) \arrow[r, shift left=2] \arrow[r, shift right=2, "{\nabla_1^{\mathbb{C}},\ \Delta^{\mathbb{C}},\ \nabla_2^{\mathbb{C}}}"'] & \mathbb{C} \arrow[l]
\end{tikzcd}$$

which is “fibrewise” a parametricity graph.

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$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{R}(\mathbb{T}) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "p^{\mathcal{R}}"'] & \mathbb{T} \arrow[l, "{\nabla_1^{\mathbb{T}},\ \Delta^{\mathbb{T}},\ \nabla_2^{\mathbb{T}}}"'] \arrow[d, "p"] \\
\mathcal{R}(\mathbb{C}) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbb{C} \arrow[l, "{\nabla_1^{\mathbb{C}},\ \Delta^{\mathbb{C}},\ \nabla_2^{\mathbb{C}}}"]
\end{tikzcd}$$

which is "fibrewise" a parametricity graph.

Note: Recover "broken" definition by dropping comprehensive.

A type theory for reasoning about parametricity

Reasoning in models: a type theory $\lambda\mathbf{2R}$

- We construct a type theory $\lambda\mathbf{2R}$ which is the 'internal language' of comprehensive $\lambda\mathbf{2}$ parametricity graphs.
- By proving soundness and completeness, we can work in $\lambda\mathbf{2R}$ instead of directly in the model.
- $\lambda\mathbf{2R}$ is similar in many respects to System R [Abadi, Cardelli and Curien, 1993] and System P [Dunphy, 2002].

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- We construct a type theory $\lambda\mathbf{2R}$ which is the 'internal language' of comprehensive $\lambda\mathbf{2}$ parametricity graphs.
- By proving soundness and completeness, we can work in $\lambda\mathbf{2R}$ instead of directly in the model.
- $\lambda\mathbf{2R}$ is similar in many respects to System R [Abadi, Cardelli and Curien, 1993] and System P [Dunphy, 2002].
- *Not* a conservative extension of $\lambda\mathbf{2}$ — parametric models enjoy much stronger properties than arbitrary models (for which $\lambda\mathbf{2}$ is internal language).

New judgement forms

$\lambda\mathbf{2R}$ extends $\lambda\mathbf{2}$ with three new judgements:

Θ rctxt	Θ is a relational context
$\Theta \vdash A_1 R A_2$ rel	R is a relation between types A_1 and A_2
$\Theta \vdash (t_1 : A_1) R (t_2 : A_2)$	$t_1 : A_1$ is related to $t_2 : A_2$ by the relation R

Relation formation rules

$$\frac{}{\Theta \vdash \alpha\rho\beta \ \mathsf{rel}} \ (\alpha\rho\beta \in \Theta) \qquad \frac{\Theta \vdash A_1 R A_2 \ \mathsf{rel} \qquad \Theta \vdash B_1 S B_2 \ \mathsf{rel}}{\Theta \vdash (A_1 \to B_1)(R \to S)(A_2 \to B_2) \ \mathsf{rel}}$$

$$\frac{\Theta,\ \alpha\rho\beta \vdash A_1 R A_2 \ \mathsf{rel}}{\Theta \vdash (\forall\alpha.\, A_1)(\forall\alpha\rho\beta.\, R)(\forall\beta.\, A_2) \ \mathsf{rel}}$$

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$$\frac{\Theta \vdash B_1 R B_2 \text{ rel} \qquad (\Theta)_1 \vdash t_1 : A_1 \to B_1 \qquad (\Theta)_2 \vdash t_2 : A_2 \to B_2}{\Theta \vdash A_1([t_1 \times t_2]^{-1} R)A_2 \text{ rel}}$$

(Will get back to projections $(-)_i$ soon.)

Direct image relations

Direct image relations

$$\frac{\Theta \vdash A_1 R A_2 \ \mathsf{rel} \qquad (\Theta)_1 \vdash t_1 : A_1 \to B_1 \qquad (\Theta)_2 \vdash t_2 : A_2 \to B_2}{\Theta \vdash B_1([t_1 \times t_2]_! R) B_2 \ \mathsf{rel}}$$

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are definable by the impredicative encoding

$$[t_1 \times t_2]_! R \coloneqq [i_{B_1} \times i_{B_2}]^{-1}(\forall \alpha \rho \beta. ([(- \circ t_1) \times (- \circ t_2)]^{-1}(R \to \rho)) \to \rho)$$

where i_B abbreviates $\lambda b. \Lambda \alpha. \lambda t. \, t \, b : B \to \forall \alpha. (B \to \alpha) \to \alpha$.

Direct image relations

Direct image relations

$$\frac{\Theta \vdash A_1 R A_2 \ \mathsf{rel} \qquad (\Theta)_1 \vdash t_1 : A_1 \to B_1 \qquad (\Theta)_2 \vdash t_2 : A_2 \to B_2}{\Theta \vdash B_1([t_1 \times t_2]_! R) B_2 \ \mathsf{rel}}$$

are definable by an impredicative encoding.

Direct image relations

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$$\frac{\Theta \vdash A_1 R A_2 \text{ rel} \qquad (\Theta)_1 \vdash t_1 : A_1 \to B_1 \qquad (\Theta)_2 \vdash t_2 : A_2 \to B_2}{\Theta \vdash B_1([t_1 \times t_2]_! R) B_2 \text{ rel}}$$

are definable by an impredicative encoding.

Semantically, this means:

Theorem

In any comprehensive $\lambda\mathbf{2}$ parametricity graph, the functors

$$\langle \nabla_1^{\mathbb{T}}, \nabla_2^{\mathbb{T}} \rangle \restriction_{\mathcal{R}(\mathbb{T})_W} : \mathcal{R}(\mathbb{T})_W \to \mathbb{T}_{\nabla_1^{\mathbb{C}} W} \times \mathbb{T}_{\nabla_2^{\mathbb{C}} W}$$

are also opfibrations (hence bifibrations).

Operations on syntax

- Left and right projections $(\cdot)_1$, $(\cdot)_2$ from relational contexts to typing contexts.

$$
\begin{aligned}
(\cdot)_i &= \cdot \\
(\Theta,\, \alpha_1 \rho \alpha_2)_i &= (\Theta)_i,\, \alpha_i \\
(\Theta,\, (x_1 : A_1) R (x_2 : A_2))_i &= (\Theta)_i,\, x_i : A_i
\end{aligned}
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\end{aligned}$$

- Conversely, a "doubling" operation takes typing contexts to relational contexts.
- Mutually defined with a "relational interpretation" $\langle A \rangle$ of types A.

$$\begin{aligned}
\langle \cdot \rangle &= \cdot & \langle \alpha \rangle &= \rho^\alpha \\
\langle \Gamma,\, \alpha \rangle &= \langle \Gamma \rangle,\, \alpha\, \rho^\alpha \alpha & \langle A \to B \rangle &= \langle A \rangle \to \langle B \rangle \\
\langle \Gamma,\, x : A \rangle &= \langle \Gamma \rangle,\, (x : A) \langle A \rangle (x : A) & \langle \forall \alpha.\, A \rangle &= \forall \alpha\, \rho^\alpha \alpha.\, \langle A \rangle
\end{aligned}$$

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\langle \Gamma,\, x : A \rangle &= \langle \Gamma \rangle,\, (x : A) \langle A \rangle (x : A) & \langle \forall \alpha.\, A \rangle &= \forall \alpha\, \rho^\alpha \alpha.\, \langle A \rangle
\end{aligned}
$$

- **Note:** Left and right hand side treated separately, so e.g. $\alpha\, \rho^\alpha \alpha$ equivalent to $\alpha\, \rho \beta$ if everything fresh.

Reflexive graph structure on syntax

Lemma

1. *If* $\Theta \vdash (t_1 : A_1)R(t_2 : A_2)$ *then* $(\Theta)_i \vdash t_i : A_i$.
2. *If* $\Gamma \vdash t : A$ *then* $\langle\Gamma\rangle \vdash (t : A)\langle A\rangle(t : A)$.

Second item is Reynolds' Abstraction Theorem in our setting.

Relatedness rules: standard relation formers

$$\frac{}{\Theta \vdash (x_1 : A_1) R (x_2 : A_2)} \; ((x_1 : A_1) R (x_2 : A_2) \in \Theta)$$

$$\frac{\Theta, (x_1 : A_1) R (x_2 : A_2) \vdash (t_1 : B_1) S (t_2 : B_2)}{\Theta \vdash (\lambda x_1.\, t_1 : A_1 \to B_1)(R \to S)(\lambda x_2.\, t_2 : A_2 \to B_2)}$$

$$\frac{\Theta \vdash (s_1 : A_1 \to B_1)(R \to S)(s_2 : A_2 \to B_2) \qquad \Theta \vdash (t_1 : A_1) R (t_2 : A_2)}{\Theta \vdash (s_1\, t_1 : B_1) S (s_2\, t_2 : B_2)}$$

$$\frac{\Theta, \alpha \rho \beta \vdash (t_1 : A_1) R (t_2 : A_2)}{\Theta \vdash (\Lambda \alpha.\, t_1 : \forall \alpha.\, A_1)\big(\forall \alpha \rho \beta.\, R\big)(\Lambda \beta.\, t_2 : \forall \beta.\, A_2)}$$

$$\frac{\Theta \vdash (t_1 : \forall \alpha.\, A_1)\big(\forall \alpha \rho \beta.\, R\big)(t_2 : \forall \beta.\, A_2) \qquad \Theta \vdash B_1 S B_2 \text{ rel}}{\Theta \vdash (t_1[B_1] : A_1[\alpha \mapsto B_1]) R[\alpha \rho \beta \mapsto B_1 S B_2](t_2[B_2] : A_2[\beta \mapsto B_2])}$$

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Relatedness rules: inverse image relations and substitution

$$\frac{\Theta \vdash (t_1\, u_1 : B_1) R (t_2\, u_2 : B_2)}{\Theta \vdash (u_1 : A_1)([t_1 \times t_2]^{-1} R)(u_2 : A_2)}$$

$$\frac{\Theta \vdash (t_1 : A_1) R (t_2 : A_2) \qquad \Theta_1 \vdash t_1 = s_1 : A_1 \qquad \Theta_2 \vdash t_2 = s_2 : A_2}{\Theta \vdash (s_1 : A_1) R (s_2 : A_2)}$$

One more rule: the parametricity rule

- The system get its power from inverse image relations together with the parametricity rule.
- Recall: If $\Gamma \vdash t : A$ then $\langle \Gamma \rangle \vdash (t : A)\langle A \rangle (t : A)$.

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- Recall: If $\Gamma \vdash s = t : A$ then $\langle \Gamma \rangle \vdash (s : A)\langle A \rangle(t : A)$.
- Parametricity rule states converse:

$$\frac{\langle \Gamma \rangle \vdash (s : A)\langle A \rangle(t : A)}{\Gamma \vdash s = t : A}$$

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- Recall: If $\Gamma \vdash s = t : A$ then $\langle \Gamma \rangle \vdash (s : A) \langle A \rangle (t : A)$.
- Parametricity rule states converse:

$$\frac{\langle \Gamma \rangle \vdash (s : A) \langle A \rangle (t : A)}{\Gamma \vdash s = t : A}$$

- So $\langle A \rangle$ is the equality relation? No! Only in closed contexts.
- In fact, for open types, $\langle A \rangle$ is not even a homogeneous relation, since $\langle \alpha \rangle = \alpha \rho \beta$.

Interpretation in comprehensive $\lambda\mathbf{2}$ parametricity graphs

$$\begin{tikzcd}
\mathcal{R}(\mathbb{T}) \arrow[r, shift left=2, "{\nabla_1^{\mathbb{T}},\ \Delta^{\mathbb{T}},\ \nabla_2^{\mathbb{T}}}"] \arrow[r, shift right=2] \arrow[d, "p^{\mathcal{R}}"'] & \mathbb{T} \arrow[l] \arrow[d, "p"] \\
\mathcal{R}(\mathbb{C}) \arrow[r, shift left=2] \arrow[r, shift right=2, "{\nabla_1^{\mathbb{C}},\ \Delta^{\mathbb{C}},\ \nabla_2^{\mathbb{C}}}"'] & \mathbb{C} \arrow[l]
\end{tikzcd}$$

- $\lambda\mathbf{2}$ interpreted in p, as before.
- Relational context Θ interpreted as an object $[\![\Theta]\!]$ in $\mathcal{R}(\mathbb{C})$.
- Syntactic relation $\Theta \vdash ARB$ rel interpreted as a semantic relation $[\![R]\!]_\Theta : [\![A]\!]_{(\Theta)_1} \leftrightarrow [\![B]\!]_{(\Theta)_2}$ in $\mathcal{R}(\mathbb{T})_{[\![\Theta]\!]}$ using $\lambda\mathbf{2}$ structure.

Interpretation of inverse image relations

- Inverse-image relation $\Theta \vdash A_1([t_1 \times t_2]^{-1} R) A_2$ rel interpreted using the *fibration* property of the parametricity graph:

- Have

$$
\begin{aligned}
[\![t_1]\!]_{(\Theta)_1} &: \mathbf{1} \longrightarrow [\![A_1]\!]_{(\Theta)_1} \Rightarrow [\![B_1]\!]_{(\Theta)_1} \\
[\![t_2]\!]_{(\Theta)_2} &: \mathbf{1} \longrightarrow [\![A_2]\!]_{(\Theta)_2} \Rightarrow [\![B_2]\!]_{(\Theta)_2}
\end{aligned}
$$

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$$[\![R]\!] : [\![B_1]\!]_{(\Theta)_1} \leftrightarrow [\![B_2]\!]_{(\Theta)_2}$$

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- Reindex $[\![R]\!] : [\![B_1]\!]_{(\Theta)_1} \leftrightarrow [\![B_2]\!]_{(\Theta)_2}$ in the fibration along these maps to interpret $[\![[t_1 \times t_2]^{-1}R]\!] : [\![A_1]\!]_{(\Theta)_1} \leftrightarrow [\![A_2]\!]_{(\Theta)_2}$.

Why didn't this work before?

- If we try to replay the interpretation in the old-fashioned semantics without comprehension, we get:

$$\begin{aligned} [\![t_1]\!]' &: ([\![\Delta]\!])_1 \longrightarrow ([\![A_1]\!])_1 \Rightarrow ([\![B_1]\!])_1 \\ [\![t_2]\!]' &: ([\![\Delta]\!])_2 \longrightarrow ([\![A_2]\!])_2 \Rightarrow ([\![B_2]\!])_2 \end{aligned}$$

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- Reindexing along this does not give a relation $([\![A_1]\!])_1 \leftrightarrow ([\![A_2]\!])_2$!

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- Reindexing along this does not give a relation $([\![A_1]\!])_1 \leftrightarrow ([\![A_2]\!])_2$!
- So things work because in the new semantics, $[\![t_i]\!]_{(\Theta)_i}$ are global points. Possible because of use of comprehension.

Soundness

Theorem (Soundness for $\lambda\mathbf{2R}$)

In every comprehensive $\lambda\mathbf{2}$ parametricity graph:

1. *if* $\Gamma \vdash t_1 = t_2 : A$ *then* $[\![t_1]\!]_\Gamma = [\![t_2]\!]_\Gamma$*; and*
2. *if* $\Theta \vdash (t_1 : A_1) R (t_2 : A_2)$ *then* $[\![t_1]\!]_{(\Theta)_1} \times [\![t_2]\!]_{(\Theta)_2} : \mathbf{1}_{[\![\Theta]\!]} \longrightarrow [\![R]\!]_\Theta$.

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2. *if* $\Theta \vdash (t_1 : A_1) R (t_2 : A_2)$ *then* $[\![t_1]\!]_{(\Theta)_1} \times [\![t_2]\!]_{(\Theta)_2} : \mathbf{1}_{[\![\Theta]\!]} \longrightarrow [\![R]\!]_\Theta$.

Substitution in relations sound by *relational* property.

Parametricity rule sound by *identity* property.

Inverse image rules sound by *fibration* property.

. . . and completeness

Theorem (Full completeness for $\lambda 2\mathbf{R}$)

There exists a comprehensive $\lambda 2$ parametricity graph satisfying the following.

1. *For every type* $\Gamma \vdash A$ type*, every global point* $\mathbf{1}_{[\![\Gamma]\!]} \longrightarrow [\![A]\!]_\Gamma$ *is the denotation* $[\![t]\!]_\Gamma$ *of some term* $\Gamma \vdash t : A$.
2. *For all terms* $\Gamma \vdash t_1, t_2 : A$ *satisfying* $[\![t_1]\!]_\Gamma = [\![t_2]\!]_\Gamma$*, we have* $\Gamma \vdash t_1 = t_2 : A$.
3. *For every relation* $\Theta \vdash A_1 R A_2$ type*, every global point* $\mathbf{1}_{[\![\Theta]\!]} \longrightarrow [\![R]\!]_\Theta$ *arises as* $[\![t_1]\!]_{(\Theta)_1} \times [\![t_2]\!]_{(\Theta)_2}$ *for terms* t_1, t_2 *such that* $\Theta \vdash (t_1 : A_1) R (t_2 : A_2)$.

Deriving the expected consequences

Warm-up: $\forall\alpha.\,\alpha \to \alpha$ is terminal

- Want to prove $\Gamma, z : \forall\alpha.\,\alpha \to \alpha \vdash z = \Lambda\alpha.\,\lambda x.\,x : \forall\alpha.\,\alpha \to \alpha$.

Warm-up: $\forall \alpha. \alpha \to \alpha$ is terminal

- Want to prove $\Gamma, z : \forall \alpha. \alpha \to \alpha \vdash z = \Lambda \alpha. \lambda x. x : \forall \alpha. \alpha \to \alpha$.
- By extensionality, it is enough to show

$$\Gamma, z : \forall \alpha. \alpha \to \alpha, \alpha, x : \alpha \vdash z[\alpha]\, x = x : \alpha$$

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- Further by the parametricity rule, it is enough to show

$$\langle \Gamma, z : \forall\alpha.\, \alpha \to \alpha, \alpha, x : \alpha \rangle \vdash (z[\alpha]\, x : \alpha)\langle\alpha\rangle(x : \alpha)$$

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- Further by the parametricity rule, it is enough to show

$$\langle\Gamma\rangle, z\big(\forall\alpha\rho\beta.\,\rho \to \rho\big)w, \alpha\rho\beta, (x : \alpha)\rho(y : \beta) \vdash (z[\alpha]\,x : \alpha)\rho(y : \beta)$$

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i.e.
$$(z[\alpha]\,x : \alpha)\rho(y : \beta).$$

$\square$

The expected consequences

Theorem (Consequences of Parametricity)

System $\lambda\mathbf{2R}$ proves:

1. $\forall\alpha.\,\alpha \to \alpha$ *is* $\mathbf{1}$.
2. $\forall\alpha.\,(A \to B \to \alpha) \to \alpha$ *is* $A \times B$.
3. $\forall\alpha.\,\alpha$ *is* $\mathbf{0}$.
4. $\forall\alpha.\,(A\to\alpha)\to(B\to\alpha)\to\alpha$ *is* $A+B$.
5. $\forall\alpha.\,(\forall\beta.\,(T(\beta) \to \alpha)) \to \alpha$ *is* $\exists\alpha.\,T(\alpha)$.
6. *The type* $\forall\alpha.\,(T(\alpha) \to \alpha) \to \alpha$ *is the carrier of the initial T-algebra for all functorial type expressions* $T(\alpha)$.
7. *The type* $\exists\alpha.\,(\alpha \to T(\alpha)) \times \alpha$ *is the carrier of the final T-coalgebra for all functorial type expressions* $T(\alpha)$.
8. *Terms of type* $\forall\alpha.\,F(\alpha,\alpha) \to G(\alpha,\alpha)$ *for mixed-variance type expressions F and G are dinatural.*

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- Subtlety: initial algebras use inverse image pseudographs, final coalgebras direct image ones.

Summary

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- $\lambda\mathbf{2}$ fibrations with comprehension property as natural models of $\lambda\mathbf{2}$ (sound and complete).
- Comprehensive $\lambda\mathbf{2}$ parametricity graphs form good models of relational parametricity for $\lambda\mathbf{2}$, with usual strong consequences.
- Reasoning in the models using a sound and complete type theory $\lambda\mathbf{2R}$, including inverse image relations.
- Proof of consequences of parametricity involves novel ingredients:
  - direct image relations via impredicative encoding,
  - no identity relations available, and
  - two different pseudo-graph relations (using inverse and direct images).
- **Future work:** Extend to e.g. dependent type theory.

Neil Ghani, Fredrik Nordvall Forsberg and Alex Simpson
Comprehensive parametric polymorphism: categorical models and type theory.
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