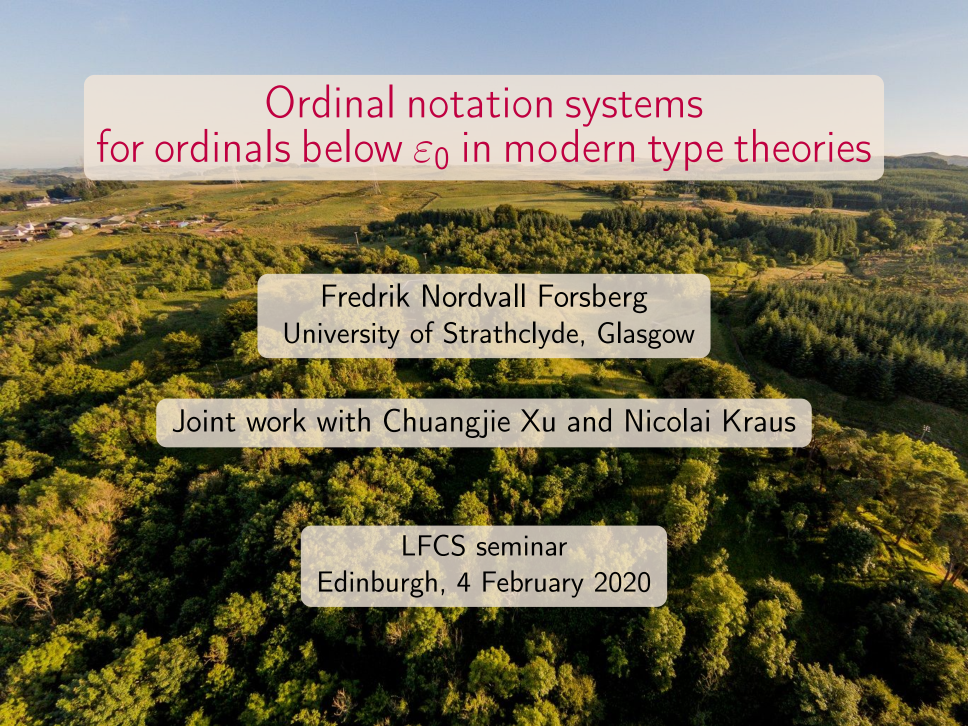

Ordinal notation systems for ordinals below ε_0 in modern type theories

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Joint work with Chuangjie Xu and Nicolai Kraus

LFCS seminar
Edinburgh, 4 February 2020

Ordinals, classically in set theory

Definition

A set α is an ordinal if it is transitive and $\in$ is well-founded on α:

- $x \in \alpha \to x \subseteq \alpha$,
- Every nonempty $X \subseteq \alpha$ has an $\in$-least element.

(Obviously too strong constructively!)

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E.g. already Turing [1949] used ordinals to prove termination of programs.

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Hence if we compute the Cantor Normal Form

$$\beta_i = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_m}$$

and so on, we get decreasing sequences

$$\alpha > \beta_i > \gamma_j > \dots$$

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which must terminate. This gives a finite representation of α!

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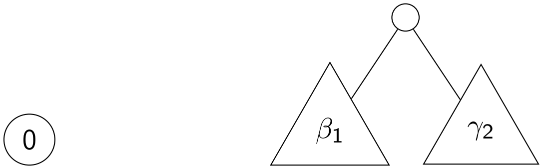
Simply binary trees! [Dershowitz 1993]



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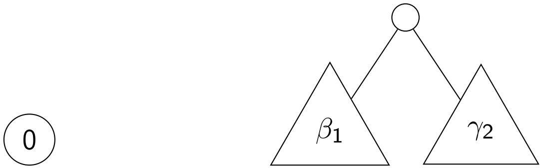

But: uniqueness of representation has been lost. How can we recover this?

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Why cubical? Want a univalence principle which computes, and higher inductive types.

A subset approach

See e.g. Buchholz [1991]

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data Tree : Type₀ where
  0 : Tree
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We single out the trees in Cantor Normal Form:

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data isCNF : Tree → Type₀ where
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This uses _≥_ : Tree→Tree→Type$_0$ (defined inductively), and

```
fst : Tree → Tree
fst 0 = 0
fst (ω^ a + _) = a
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SigmaOrd

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Pro: Not requiring any fancy features.

Con: "Junk terms". Code duplication.


A mutual approach

Intrinsically Cantor Normal Form ordinals

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We simultaneously define

$$\begin{aligned}
&\mathsf{data}\ \mathsf{MutualOrd} : \mathsf{Type}_0 \\
&\mathsf{data}\ _<_ : \mathsf{MutualOrd} \to \mathsf{MutualOrd} \to \mathsf{Type}_0 \\
&\mathsf{fst} : \mathsf{MutualOrd} \to \mathsf{MutualOrd}
\end{aligned}$$

by induction-induction-recursion [N.-F. 2014].

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data _<_ where
  <₁ : 0 < ω^ a + b [ r ]
  <₂ : a < c → ω^ a + b [ r ] < ω^ c + d [ s ]
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Remark: there is an equivalent non-inductive-recursive definition where we define the graph of fst inductively.

Examples

- $\mathbf{0}$
- $\mathbf{1} = \omega\hat{}\ \mathbf{0} + \mathbf{0}\ [\ \mathsf{inj}_2\ \mathsf{refl}\ ]$
- $\omega = \omega\hat{}\ \mathbf{1} + \mathbf{0}\ [\ \mathsf{inj}_1 <_1\ ]$
- $\omega\hat{}\langle\ a\ \rangle = \omega\hat{}\ a + \mathbf{0}\ [\ \geq\mathbf{0}\ ]$

Basic properties

Proposition

$\_<\_$ *is proof-irrelevant, i.e.* $p \equiv q$ *for any* $p, q : a < b$*.*

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Theorem

Transfinite induction holds for MutualOrd, i.e. there is a proof

$$\begin{aligned} MTI &: (P : MutualOrd \to Type\ \ell) \\ &\to (\forall\ x \to (\forall\ y \to y < x \to P\ y) \to P\ x) \\ &\to \forall\ x \to P\ x \end{aligned}$$

Not provable without unique representation!

Ordinal addition

Addition on ordinals is famously non-commutative

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We now want to implement addition on MutualOrd. We simultaneously define

$$\begin{aligned} \_+\_ &: \mathsf{MutualOrd} \to \mathsf{MutualOrd} \to \mathsf{MutualOrd} \\ {\geq}\mathsf{fst}{+} &: \{a : \mathsf{MutualOrd}\}\ (b\ c : \mathsf{MutualOrd}) \\ &\to a \geq \mathsf{fst}\ b \to a \geq \mathsf{fst}\ c \to a \geq \mathsf{fst}\ (b + c) \end{aligned}$$

Addition on MutualOrd

Remember: if $\alpha < \beta$ then $\omega^{\alpha} + \omega^{\beta} = \omega^{\beta}$.

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$\mathbf{0} + b = \{?_0 : \text{MutualOrd}\}$

$a + \mathbf{0} = \{?_1 : \text{MutualOrd}\}$

$(\omega\hat{}\ a + c\,[\,r\,]) + (\omega\hat{}\ b + d\,[\,s\,]) = \{?_2 : \text{MutualOrd}\}$

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```
0 + b = b
a + 0 = a
(ω^ a + c [ r ]) + (ω^ b + d [ s ])  with <-tri a b
... | inj₁ a<b = {?₂ : MutualOrd}
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```

Addition on MutualOrd

Remember: if $\alpha < \beta$ then $\omega^\alpha + \omega^\beta = \omega^\beta$.

```
0 + b = b
a + 0 = a
(ω^ a + c [ r ]) + (ω^ b + d [ s ])  with <-tri a b
... | inj₁ a<b = ω^ b + d [ s ]
... | inj₂ a≥b = {?₃ : MutualOrd}
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≥fst+ 0 _ r s = s
≥fst+ (ω^ _ + _ [ _ ]) 0 r s = r
≥fst+ (ω^ b + _ [ _ ]) (ω^ c + _ [ _ ]) r s with <-tri b c
... | inj₁ b<c = s
... | inj₂ b≥c = r
```

Multiplication on MutualOrd

$$
\begin{aligned}
&\_\cdot\_ : \mathsf{MutualOrd} \to \mathsf{MutualOrd} \to \mathsf{MutualOrd} \\
&\mathbf{0} \cdot b = \mathbf{0} \\
&a \cdot \mathbf{0} = \mathbf{0} \\
&a \cdot (\omega\hat{}\ \mathbf{0} + d\,[\,r\,]) = a + a \cdot d \\
&(\omega\hat{}\ a + c\,[\,r\,]) \cdot (\omega\hat{}\ b + d\,[\,s\,]) = \\
&\quad \mathsf{M}.\omega\hat{}\langle\, a + b \,\rangle + (\omega\hat{}\ a + c\,[\,r\,] \cdot d)
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\end{aligned}
$$

Note: All in terms of previous operations, so no simultaneous lemma needed.


A higher inductive approach

Uniqueness by making things the same

We want to avoid redundant representations of ordinals

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n}$$

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Cubical Agda allows this via higher inductive types [Lumsdaine and Shulman 2019].

A higher inductive approach

Inspired by Licata's [2014] encoding of finite multisets [Blanchette, Fleury and Traytel 2017] as a HIT

A Higher Inductive Type also allows constructors targetting equalities between elements (and between equalities, equalities between equalities, . . .).

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We define:

```
data HITOrd : Type₀ where
  𝟎 : HITOrd
  ω^_⊕_ : HITOrd → HITOrd → HITOrd
  swap : ∀ a b c → ω^ a ⊕ ω^ b ⊕ c ≡ ω^ b ⊕ ω^ a ⊕ c
  trunc : isSet HITOrd
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```

$p \equiv q$ for all $p, q : a \equiv_{\mathsf{HITOrd}} b$

Example

```
example : (a b c : HITOrd)
  → ω^ a ⊕ ω^ b ⊕ ω^ c ⊕ 0 ≡ ω^ c ⊕ ω^ b ⊕ ω^ a ⊕ 0
example a b c = begin
  ω^ a ⊕ ω^ b ⊕ ω^ c ⊕ 0 ≡⟨ swap a b _ ⟩
  ω^ b ⊕ ω^ a ⊕ ω^ c ⊕ 0 ≡⟨ cong (ω^ b ⊕_) (swap a c _) ⟩
  ω^ b ⊕ ω^ c ⊕ ω^ a ⊕ 0 ≡⟨ swap b c _ ⟩
  ω^ c ⊕ ω^ b ⊕ ω^ a ⊕ 0 ∎
```

Pattern matching on HITOrd

Pattern matching on HITOrd requires all functions f to respect swap: must show

$$f\ (\omega\hat{}\ a \oplus \omega\hat{}\ b \oplus c) \equiv f\ (\omega\hat{}\ b \oplus \omega\hat{}\ a \oplus c)$$

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For arithmetic, these are the so-called Hessenberg sum and product [Hessenberg, 1906].

Hessenberg sum

$$\_\oplus\_ : \mathsf{HITOrd} \to \mathsf{HITOrd} \to \mathsf{HITOrd}$$
$$x \oplus y = \boxed{\{?_0 : \mathsf{HITOrd}\}}$$

Hessenberg sum

$$
\begin{array}{l}
\quad \_\oplus\_ : \mathsf{HITOrd} \to \mathsf{HITOrd} \to \mathsf{HITOrd} \\
\mathbf{0} \qquad\qquad\qquad\quad\; \oplus\, y = \{?_0 : \mathsf{HITOrd}\} \\
(\omega\hat{}\ a \oplus b) \qquad \oplus\, y = \{?_1 : \mathsf{HITOrd}\} \\
(\mathsf{swap}\ a\ b\ c\ i) \oplus y = \{?_2 : \ldots \equiv \ldots\}\ i \\
(\mathsf{trunc}\ p\ q\ i\ j) \oplus y = \{?_3 : \ldots \equiv_{\ldots \equiv \ldots} \ldots\}\ i\ j
\end{array}
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Hessenberg sum

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\begin{aligned}
&\_\oplus\_ : \mathsf{HITOrd} \to \mathsf{HITOrd} \to \mathsf{HITOrd} \\
&\mathbf{0} \quad\quad\quad\quad\quad\; \oplus\ y = y \\
&(\omega\hat{}\ a \oplus b) \quad\; \oplus\ y = \omega\hat{}\ a \oplus (b \oplus y) \\
&(\mathsf{swap}\ a\ b\ c\ i) \oplus y = \boxed{\{?_2 : \ldots \equiv \ldots\}}\ i \\
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In the swap case, we have to prove

$$?_2 : \omega\hat{}\ a \oplus \omega\hat{}\ b \oplus (c \oplus y) \equiv \omega\hat{}\ b \oplus \omega\hat{}\ a \oplus (c \oplus y)$$

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Proposition

$\_\oplus\_$ *is commutative.*

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Depending on the application, e.g. the mutual approach for properties of the order, the HIT approach for commutative operations.

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Even better:

Theorem
SigmaOrd, MutualOrd and HITOrd are equivalent.

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All of them!

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Even better:

Theorem
SigmaOrd, MutualOrd and HITOrd are equivalent.

Using the univalance principle [Voevodsky 2010] (which computes in cubical Agda), equivalent types are identical:

Corollary
SigmaOrd, MutualOrd and HITOrd are identical.

MutualOrd and HITOrd are equivalent

$$\begin{tikzcd} \mathsf{MutualOrd} \arrow[rr, bend right] & & \mathsf{HITOrd} \arrow[ll, bend right] \end{tikzcd}$$

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$\mathsf{M{\equiv}H : MutualOrd \equiv HITOrd}$

Operations via univalence

By using univalence, we can transport operations and proofs between MutualOrd and HITOrd.

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$$\_<^{\mathsf{H}}\_ : \mathsf{HITOrd} \to \mathsf{HITOrd} \to \mathsf{Type}_0$$
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$$
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$$

Transporting proofs

We can also transport properties. For instance: define

$$\mathsf{Dec} : (A : \mathsf{Type}\ \ell) \to (A \to A \to \mathsf{Type}\ \ell') \to \mathsf{Type}\ (\ell \sqcup \ell')$$
$$\mathsf{Dec}\ A\ \_{<}\_ = (x\ y : A) \to x < y \uplus \neg\, x < y$$

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Hence we can construct

$$<^{\mathsf{H}}\text{-dec} : \mathsf{Dec}\ \mathsf{HITOrd}\ \_<^{\mathsf{H}}\_$$
$$<^{\mathsf{H}}\text{-dec} = \mathsf{transport}\ (\lambda\ i \to \mathsf{Dec}\ (\mathsf{M{\equiv}H}\ i)\ ({<}\mathsf{Path}\ i))\ <\text{-dec}$$

where

$$<\mathsf{Path} : \mathsf{PathP}\ (\lambda\ i \to \mathsf{M{\equiv}H}\ i \to \mathsf{M{\equiv}H}\ i \to \mathsf{Type}_0)\ \_<\_\ \_<^{\mathsf{H}}\_$$

is a dependent equality ("path") between $\_<\_$ and $\_<^{\mathsf{H}}\_$.

It computes!

Define

$$\mathsf{lt} : \mathsf{HITOrd} \to \mathsf{HITOrd} \to \mathsf{Bool}$$
$$\mathsf{lt}\ a\ b = \mathsf{isLeft}\ (<^{\mathsf{H}}\text{-}\mathsf{dec}\ a\ b)$$

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$$\begin{aligned}
&\mathsf{Ex}[{<^{\mathsf{H}}}\text{-}\mathsf{decComp}] : \\
&\quad\quad \mathsf{lt}\ \mathbf{0}\ \mathbf{0} \equiv \mathsf{false} \\
&\quad \times\ \mathsf{lt}\ \mathsf{H}.\omega\ ((\mathsf{H}.\mathbf{1} \oplus \mathsf{H}.\mathbf{1}) \otimes \mathsf{H}.\omega) \equiv \mathsf{true} \\
&\quad \times\ \mathsf{lt}\ (\mathsf{H}.\omega\hat{}\langle\ \mathsf{H}.\omega\ \rangle)\ (\mathsf{H}.\omega\hat{}\langle\ \mathsf{H}.\mathbf{1} +^{\mathsf{H}} \mathsf{H}.\omega\ \rangle) \equiv \mathsf{false} \\
&\quad \times\ \mathsf{lt}\ (\mathsf{H}.\omega\hat{}\langle\ \mathsf{H}.\omega\ \rangle)\ (\mathsf{H}.\omega\hat{}\langle\ \mathsf{H}.\mathbf{1} \oplus \mathsf{H}.\omega\ \rangle) \equiv \mathsf{true} \\
&\mathsf{Ex}[{<^{\mathsf{H}}}\text{-}\mathsf{decComp}] = (\mathsf{refl}\ ,\ \mathsf{refl}\ ,\ \mathsf{refl}\ ,\ \mathsf{refl})
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\end{aligned}$$

$$\mathsf{Ex}[\oplus^{\mathsf{M}}\mathsf{Comp}] : \mathsf{M}.\mathbf{1} \oplus^{\mathsf{M}} \mathsf{M}.\omega \equiv \mathsf{M}.\omega + \mathsf{M}.\mathbf{1}$$
$$\mathsf{Ex}[\oplus^{\mathsf{M}}\mathsf{Comp}] = \mathsf{refl}$$

Summary and outlook

Conclusions

- Summary: Using mutual definitions and higher inductive types to faithfully represent ordinals in cubical Agda.

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- Moral: Define operations on the data structure that is suited for the operation (then transport across with univalence).
- Future work: Going beyond ε_0 using a higher inductive type of Brouwer ordinals.

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CPP 2020, New Orleans, USA.

Conclusions

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Thanks!

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