Compositional Game Theory in Type Theory

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TYPES 2019, Oslo, 14 June

What is economic game theory?

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The theory of interacting "rational" agents.

Players make observations and then make choices.

Choices of all players determine payoffs.

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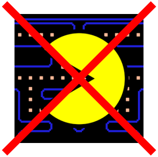

Example: penalty shootout

Choices $\Sigma = \{L, R\}^2$.

Payoffs $u : \Sigma \to \mathbb{R}^2$ with $u(a,b) = \begin{cases} (\;\;1, -1) & \text{if } a \neq b \\ (-1, \;\;1) & \text{if } a = b \end{cases}$

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No (deterministic) equilibria.

The problem of scaling

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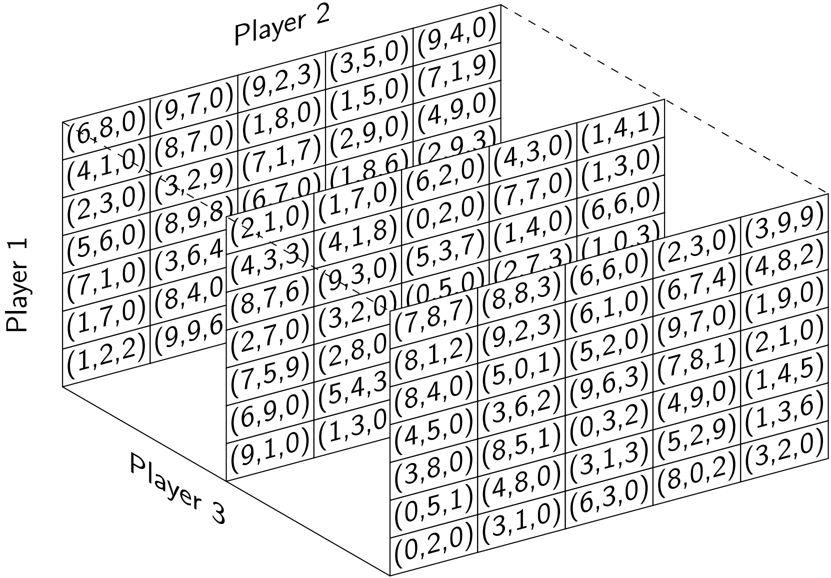
Player 2

Player 1

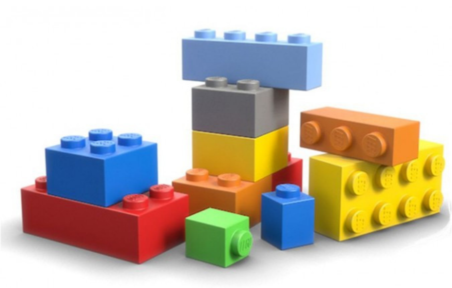
(5,2)	(1,0)	(4,0)	(3,6)	(8,1)	(0,9)	(2,5)	(0,4)	(6,3)	(8,0)
(6,2)	(3,1)	(2,5)	(8,3)	(5,8)	(6,0)	(3,8)	(9,6)	(6,5)	(8,2)
(8,5)	(9,7)	(3,6)	(8,1)	(4,7)	(2,0)	(0,6)	(2,9)	(0,4)	(5,2)
(6,2)	(3,1)	(4,0)	(7,7)	(2,7)	(0,7)	(7,1)	(9,5)	(3,8)	(6,7)
(1,8)	(9,2)	(5,9)	(2,1)	(2,2)	(8,2)	(8,6)	(1,4)	(0,2)	(0,7)
(5,9)	(8,4)	(5,8)	(1,8)	(2,7)	(0,2)	(7,1)	(2,6)	(6,3)	(0,0)
(8,8)	(0,1)	(9,1)	(3,5)	(5,8)	(6,7)	(2,9)	(6,9)	(8,2)	(3,4)
(7,9)	(6,9)	(5,7)	(4,7)	(7,0)	(3,8)	(5,8)	(9,2)	(7,1)	(8,3)
(3,9)	(6,4)	(7,7)	(5,4)	(1,7)	(9,0)	(4,8)	(4,9)	(6,4)	(0,5)
(1,2)	(1,2)	(3,5)	(6,3)	(9,3)	(2,9)	(5,2)	(8,7)	(0,3)	(5,1)
(9,1)	(0,1)	(8,8)	(2,4)	(4,6)	(1,0)	(6,0)	(2,6)	(5,7)	(3,9)

The problem of scaling



Building games compositionally

Goal: Instead of making sense of large games *post facto*, construct them from smaller, already understood games.



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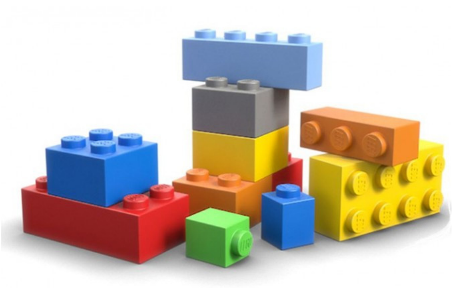

Building games compositionally

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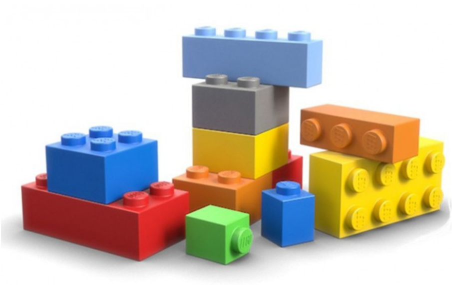
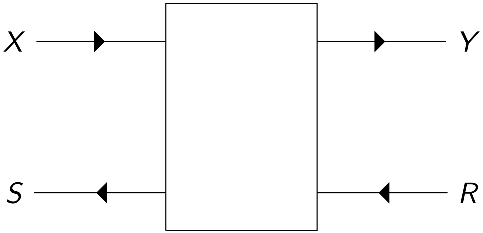
Methods: Category theory (for compositionality), type theory (for precision; this work).




The open games framework

Open games [Hedges 2016]

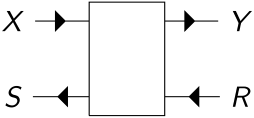
From the outside



$X \in \mathrm{Set}$	state of the game	$Y \in \mathrm{Set}$	moves of the game
$S \in \mathrm{Set}$	coutility type	$R \in \mathrm{Set}$	utility type

Open games

Inside the box



Open games

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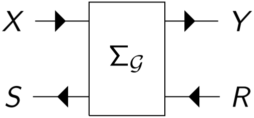
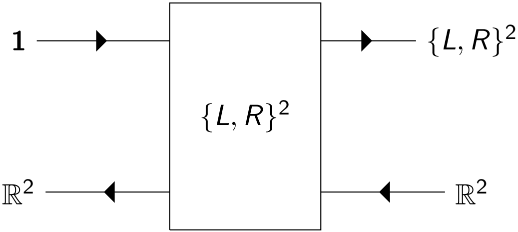


Definition

An open game $\mathcal{G} = (\Sigma_{\mathcal{G}}, P_{\mathcal{G}}, C_{\mathcal{G}}, E_{\mathcal{G}}) : (X, S) \to (Y, R)$ consists of:

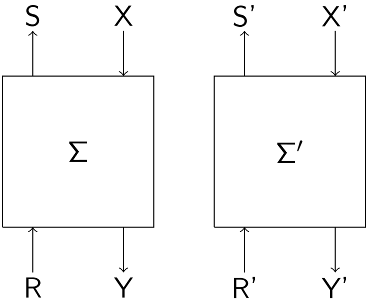
- a set $\Sigma_{\mathcal{G}}$ of strategy profiles,
- a play function $P_{\mathcal{G}} : X \to \Sigma_{\mathcal{G}} \to Y$,
- a coutility function $C_{\mathcal{G}} : X \to \Sigma_{\mathcal{G}} \to R \to S$, and
- an equilibrium function $E_{\mathcal{G}} : X \to (Y \to R) \to \mathscr{P}(\Sigma_{\mathcal{G}})$.

Example: penalty shootout as an open game

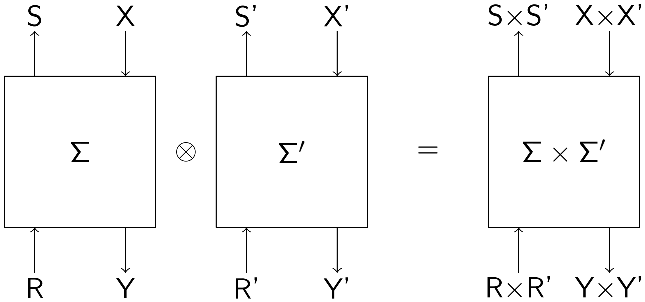


$$\begin{aligned}
P(x, \sigma) &= \sigma \\
C(x, \sigma, r) &= r \\
(a, b) \in E(x, k) \text{ iff } & \pi_1(k(a, b)) \geq \pi_1(k(\bar{a}, b)) \text{ and} \\
& \pi_2(k(a, b)) \geq \pi_2(k(a, \bar{b}))
\end{aligned}$$

Parallel composition of open games



Parallel composition of open games



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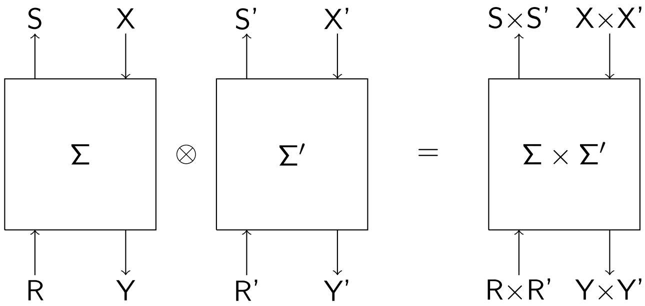


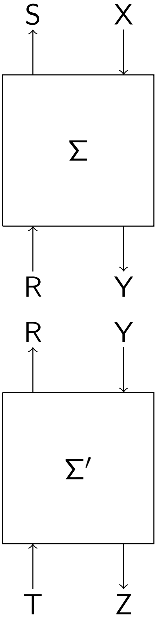
Proposition

The penalty shootout open game can be built as $P_1 \otimes P_2$, where

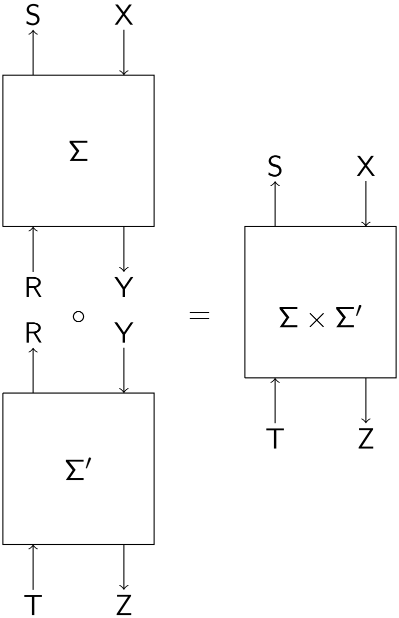
$$P_1, P_2 : (\mathbf{1}, \mathbb{R}) \to (\{L, R\}, \mathbb{R})$$

with $\Sigma_{P_i} = \{L, R\}$, and $a \in E_{P_i}(x, k)$ iff $a \in \arg\max_{x \in \Sigma}\{k(x)\}$.

Sequential composition



Sequential composition



Symmetric monoidal structure

Theorem ([Ghani, Hedges, Winschel, Zahn 2018])

(i) *The collection of pairs of sets, with open games* $\mathcal{G} : (X, S) \to (Y, R)$ *as morphisms, forms a symmetric monoidal category* Game.

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Theorem ([Ghani, Hedges, Winschel, Zahn 2018])

(i) *The collection of pairs of sets, with open games $\mathcal{G} : (X, S) \to (Y, R)$ as morphisms, forms a symmetric monoidal category* Game*.*

(ii) *There is a identity-on-objects functor*

$$\iota : \mathsf{Set} \times \mathsf{Set}^{\mathsf{op}} \to \mathsf{Game}$$

with

$$P_{\iota(f,g)}(x, \sigma) = f(x) \qquad\qquad C_{\iota(f,g)}(x, \sigma, r) = g(r). \quad \square$$

More structure?

Can we construct e.g. coproducts of games? (For a natural notion of morphisms between games.)

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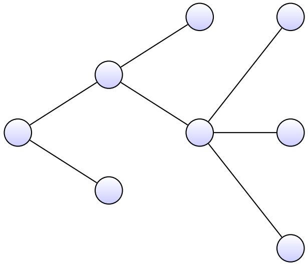
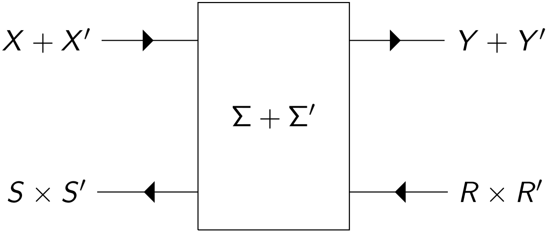
Game-theoretic motivation: Games with *external* choice, e.g. later rounds depend on choices in previous rounds.



Coproduct construction attempts

First attempt:



$$P_{\mathcal{G}+\mathcal{G}'} : (X + X') \times (\Sigma + \Sigma') \to (Y + Y')$$

$$P_{\mathcal{G}+\mathcal{G}'}(\mathsf{inl}\, x)\,(\mathsf{inl}\,\sigma) = \{?_0 : Y + Y'\}$$

$$P_{\mathcal{G}+\mathcal{G}'}(\mathsf{inl}\, x)\,(\mathsf{inr}\,\sigma') = \{?_1 : Y + Y'\}$$

$$\vdots$$

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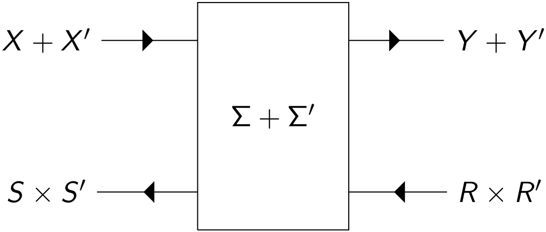

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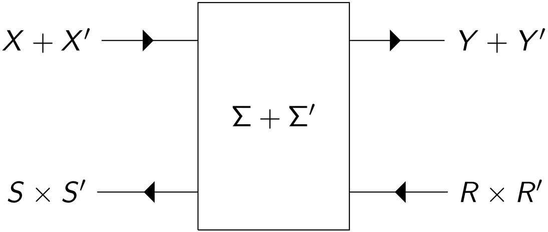

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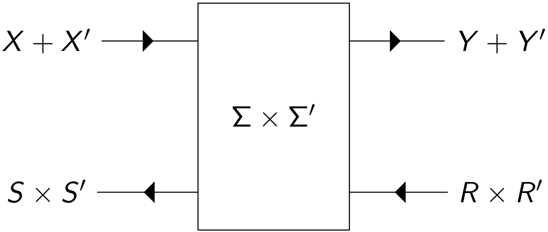
$$\vdots$$

Coproduct construction attempts

~~First~~ Second attempt:



$$P_{\mathcal{G}+\mathcal{G}'} : (X + X') \times (\Sigma \times \Sigma') \to (Y + Y')$$

$$P_{\mathcal{G}+\mathcal{G}'}(\mathsf{inl}\, x)\,(\sigma, \sigma') = \{?_0 : Y + Y'\}$$

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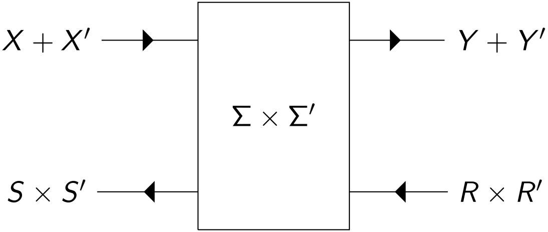

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&P_{\mathcal{G}+\mathcal{G}'}(\mathsf{inr}\, x)\,(\sigma, \sigma') = \boxed{\{?_1 : Y + Y'\}}
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$$

$$\vdots$$

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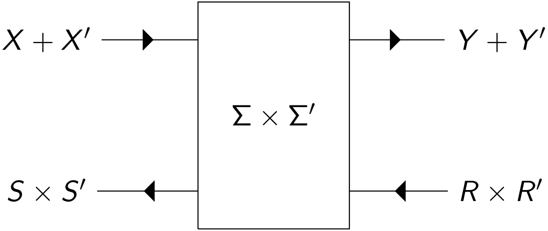

$$
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P_{\mathcal{G}+\mathcal{G}'} &: (X + X') \times (\Sigma \times \Sigma') \to (Y + Y') \\
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P_{\mathcal{G}+\mathcal{G}'}(\mathsf{inr}\, x)\,(\sigma, \sigma') &= \mathsf{inr}\,(P_{\mathcal{G}'}\, x\, \sigma') \\
&\vdots
\end{aligned}
$$

But: To define injections $\mathcal{G} \to \mathcal{G} + \mathcal{G}'$ we need a strategy component $\Sigma_G \to \Sigma_G \times \Sigma'_G$. ↯

Analysis

We kept both strategies around because we could not describe the situations when we needed one but not the other.

(This is reminiscent of implementing $A + B$ as $A \times B$, and supplying a dummy value as needed.)

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But. . . what if we could be more precise about which strategy we need?

Open games in type theory

Introducing dependency

Old definition:

$$
\begin{aligned}
X &: \mathsf{Set} \\
S &: \mathsf{Set} \\
Y &: \mathsf{Set} \\
R &: \mathsf{Set} \\
\Sigma &: \mathsf{Set} \\
P &: X \to \Sigma \to Y \\
C &: X \to \Sigma \to R \to S \\
E &: X \to (Y \to R) \to \mathscr{P}(\Sigma)
\end{aligned}
$$

Introducing dependency

Dependently typed definition:

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Y &: \mathsf{Set} \\
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\Sigma &: X \to \mathsf{Set} \\
P &: (x : X) \to \Sigma\, x \to Y \\
C &: (x : X) \to (\sigma : \Sigma\, x) \to R\,(P\, x\, \sigma) \to S\, x \\
E &: (x : X) \to ((y : Y) \to R\, y) \to \mathscr{P}(\Sigma\, x)
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\end{aligned}
$$

Note: (X, S) is a *container* [Abbott, Altenkirch, Ghani 2005].

Dependently typed open games

Let (X, S) and (Y, R) be containers.

Definition

A dependently typed open game $\mathcal{G} : (X, S) \to (Y, R)$ consists of:

- a family of sets $\Sigma_{\mathcal{G}} : X \to \mathrm{Set}$,
- a play function $P_{\mathcal{G}} : (x : X) \to \Sigma_{\mathcal{G}}(x) \to Y$,
- a coutility function
  $C_{\mathcal{G}} : (x : X) \to (\sigma : \Sigma_{\mathcal{G}}) \to R(P_{\mathcal{G}}\, x\, \sigma) \to S(x)$, and
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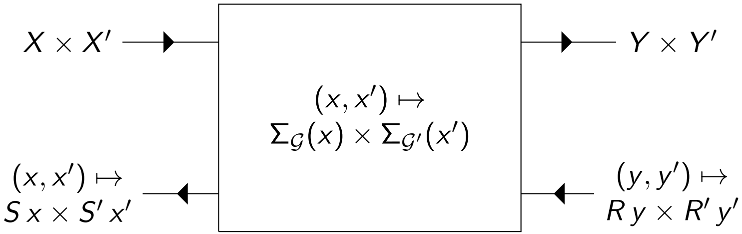
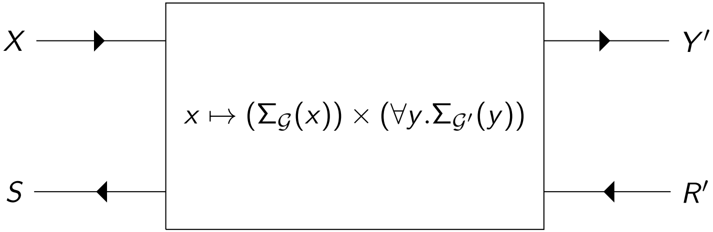
- a equilibrium function $E_\mathcal{G} : (x : X) \to ((y : Y) \to R(y)) \to \mathscr{P}(\Sigma_\mathcal{G}(x))$.

Observation: If S, R, $\Sigma_\mathcal{G}$ are constant families, this reduces to an ordinary open game.

Parallel composition of dependently typed games



Sequential composition of dependently typed games



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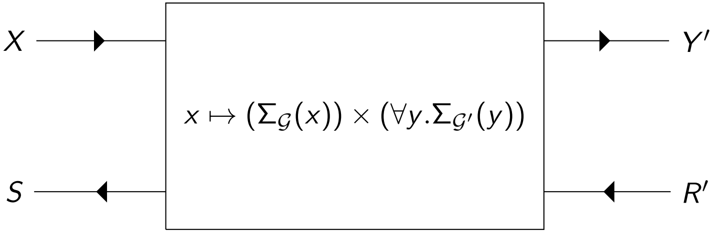

Note: "Alternative" definition

$$\Sigma_{\mathcal{G}' \circ \mathcal{G}}\, x = (\sigma : \Sigma_{\mathcal{G}}(x)) \times (\Sigma_{\mathcal{G}'}\, (P_{\mathcal{G}}\, x\, \sigma))$$

does not work.

Uniform function space $\forall y.B(y)$

Intuitively, consists of functions that make no computational use of their argument. (cf. “ghost variables” in Hoare logic).

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Modelled by intersection in PER/realizability models.

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Modelled by intersection in PER/realizability models.

In Agda: run-time irrelevance `@0` + Frobenius axiom

$$\forall x.(B \times P(x)) \cong B \times \forall x.P(x)$$

Symmetric monoidal structure

Theorem

(i) *The collection of containers, with open games* $\mathcal{G} : (X, S) \to (Y, R)$ *as morphisms, forms a symmetric monoidal category* DGame.

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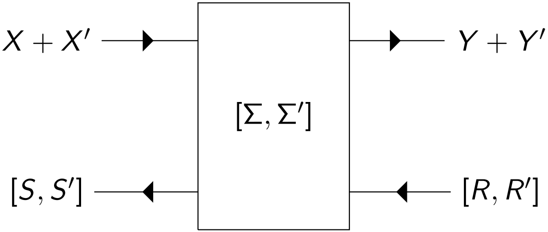
(ii) *There is a identity-on-objects functor*

$$\iota : \mathrm{Cont} \to \mathrm{DGame}$$

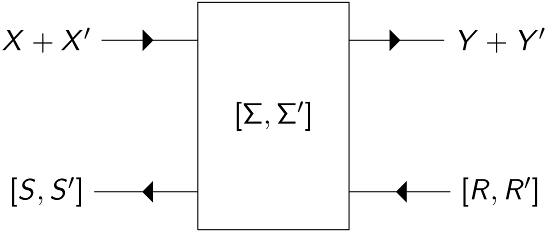
$\square$

Coproducts of dependently typed games



$$P_{\mathcal{G}+\mathcal{G}'}\,(\mathsf{inl}\,x, \sigma) = \mathsf{inl}\,(P_{\mathcal{G}}(x, \sigma))$$
$$P_{\mathcal{G}+\mathcal{G}'}\,(\mathsf{inr}\,x', \sigma') = \mathsf{inr}\,(P_{\mathcal{G}'}(x', \sigma'))$$

Coproducts of dependently typed games



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Also has the right universal property.

Summary

Compositional Game Theory in Type Theory

- Open games as a compositional model of game theory.
- Dependently typed open games for more precision in the model, and a mathematically nicer category of games (e.g. coproducts of games).

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Scottish Programming Languages and Verification Summer School

5–9 August 2019, Glasgow, Scotland

`http://www.macs.hw.ac.uk/splv/splv19/`

Chung-chieh Shan	Probabilistic programming
Phil Wadler	Programming Language Foundations in Agda
Neil Ghani	Category Theory
Conor McBride	Dependently Typed Programming
Ornela Dardha	Session types
Greg Michaelson, Rob Stewart	Domain-specific languages
Chris Brown	Parallel Programming